$$\frac{T_2^2}{r_2^3} = \frac{4\pi^2}{GM},$$

where T_2 and r_2 are the period and orbit radius respectively, for the second planet.

From last two results we obtain

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{r_1}{r_2}\right)^3,$$

which is Kepler's third law.

4. WORK AND ENERGY

The present chapter is devoted to the very important concept of energy and the closely related concept of work. These two quantities are scalars and so have no direction. Energy derives its importance from two sources. First, it is a conserved quantity and second, energy is a concept that is useful not only in the study of motion but in all areas of physics and in other sciences as well.

4 - 1 Work done by a Constant Force

Definition: the work done on a particle by a constant force (constant in both magnitude and direction) is defined as the product of the magnitude of the displacement times the component of the force parallel to the displacement.

So,

$$W = F\,x\cos\varphi, \qquad (4\text{-}1)$$

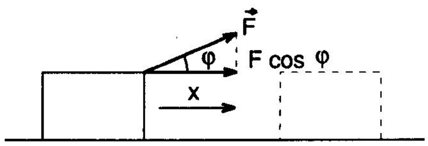

Figure 4 - 1

where F is the constant force, x is the net displacement of the particle and φ is the angle between the directions of the force and the net displacement. Let us note the factor $F\cos\varphi$ is the component of the force parallel to the displacement (see Fig. 4-1).

Simple case occurs when the motion and the force are in the same direction, so that $\cos\varphi = 1$ and $W = Fd$.

The unit of work is the joule (J): J = N m.

Example : A box of mass m is pulled on distance x along a horizontal floor by a constant force F_p which acts at angle φ. The floor is rough and exerts a friction force F_f.

Determine the work done by each force acting on the box, and the net work done on the box.

Solution: There are four forces acting on the box as shown in Fig. 4-2; the pulled force F_p, the friction force F_f, the weight of box mg and the normal force F_N exerted upward by the floor.

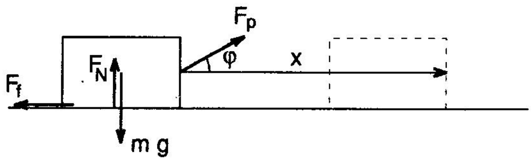

Figure 4-2

The work done by the gravitational and normal forces is zero since they are perpendicular to the displacement:

$$W_g = 0$$

$$W_N = 0$$

The work done by F_p is

$$W_p = F_p x \cos \varphi .$$

The work done by friction force is

$$W_f = F_f x \cos \pi ,$$

so the force of friction does negative work on the box.

The net work done on an object we determine as the sum of the work done by each force:

$$W = W_g + W_N + W_p + W_f = (F_p \cos \varphi + F_f \cos \pi)x .$$

4 - 2 Work done by a Nonconstant Force

In many cases a force varies its magnitude or direction. We now calculate the work done by a nonconstant force.

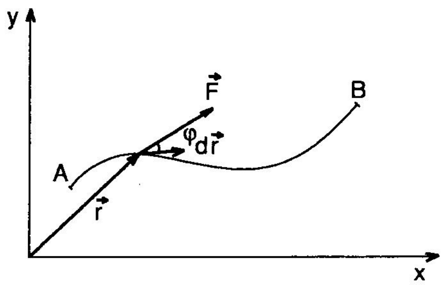

Figure 4-3

Let us assume a particle moving from point A to point B along the path as figure 4-3 shows. The path can be divided into infinitesimal intervals. During each such interval the force is assumed to be approximately constant. If $\vec{r}$ is the position vector of any infinitesimal interval, this interval can be described by the infinitesimal displacement vector $d\vec{r}$. The direction of the vector $d\vec{r}$ is along the tangent to the curve at that point which has the position vector $\vec{r}$. So, φ is the angle between $\vec{F}$ and $d\vec{r}$.

If we denote $d\ell = |d\vec{r}|$ as the magnitude of the infinitesimal displacement vector $d\vec{r}$, the work done by the force $\vec{F}$ along the infinitesimal interval of the path is with respect to Eq. 4-1 equal

$$dW = F\, d\ell \cos \varphi .$$

A net work done along the path from point A to B equals

$$W = \int_A^B F \cos \varphi \, d\ell = \int_A^B \vec{F}\, d\vec{r} , \qquad (4-2)$$

where we use scalar product notation.

Note that the factor $F \cos \varphi$ represents the component of the force $\vec{F}$ parallel to the curve tangent at any point. So, only the component of $\vec{F}$ parallel to the velocity vector, $F \cos \varphi$, contributes to work. A force or component of a force acting perpendicular to the velocity vector does no work. Such a force changes only the direction of the velocity, it does not affect the magnitude of the velocity. Example of this is uniform circular motion with constant speed. Centripetal force does no work on the object.

Eq. 4-2 is the most general definition of work. Note that work is a scalar quantity. To calculate the integral in Eq. 4-2 we must be able to express a non-constant force $\vec{F}$ as a function of position.

4 - 3 Power

Power is defined as the rate at which work is done. The average power $\bar{P}$, when an amount of work W is done in time t is

$$\bar{P} = \frac{W}{t} . \quad (4\text{-}3)$$

The instantaneous power P is

$$P = \frac{dW}{dt} , \quad (4\text{-}4)$$

where dW is elementary work done in element time dt. In SI units power is measured in joules per second called watt (W) = J/s.

It is often convenient to write the power in terms of the net force $\vec{F}$ applied to an object and its velocity $\vec{v}$. Since $P = dW/dt$ and $dW = \vec{F}.d\vec{\ell}$, then

$$P = \frac{dW}{dt} = \vec{F}\frac{d\vec{\ell}}{dt} = \vec{F}\ \vec{v} . \quad (4\text{-}5)$$

4 - 4 Kinetic Energy

Energy is one of the most important concepts in science. There are various types of energy. In this section we define kinetic energy of the translational motion.

Suppose the net force $\vec{F}$ on an object varies in both magnitude and direction, and the path of the object is a curve as in Fig. 4-3. The force may be considered to be a function of ℓ , the distance along the curve. The work done by this force is (Eq. 4-2):

$$W = \int F \cos\varphi \, d\ell . \quad (4\text{-}6)$$

Since $F\cos\varphi$ represents the component of the force $\vec{F}$ parallel to the curve tangent at any point, by the second law of motion we may write

$$F\cos\varphi = m\, a_t , \quad (4\text{-}7)$$

where a_t is the component of acceleration a parallel to the curve tangent at any point (so, tangential acceleration), which is equal to the rate of change of speed, dv/dt. We can assume v as a function of ℓ , and using the chain rule for derivatives:

$$\frac{dv}{dt} = \frac{dv}{d\ell}\frac{d\ell}{dt} = v\frac{dv}{d\ell} ,$$

since $d\ell/dt$ is the speed v .

Now the equation (4-7) may be rewritten in form

$$F\cos\varphi = m\frac{dv}{dt} = mv\frac{dv}{d\ell} ,$$

and after substituting into Eq. 4-3 we have:

$$W = \int_{v_1}^{v_2} mv\, dv = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2 = \Delta KE , \quad (4\text{-}8)$$

where v_1, v_2 are initial and final speed, respectively, and ΔKE represents the change in kinetic energy whose instantaneous magnitude is $KE = \frac{1}{2} mv^2$. The equation (4-8) states:

> The net work done on an object is equal to its change in kinetic energy.

This is known as the <u>work-energy theorem</u>. It tell us that if positive work W is done on a body, its kinetic energy increases by amount W or if negative work W is done on the body, its kinetic energy decreases by amount W . And if work done on the body is zero, its kinetic energy remains constant.

Energy is measured in the same units as work: joule (J). Like work, kinetic energy is a scalar quantity.

4 - 5 Conservative and Nonconservative Forces

It is useful to divide forces into two types:

> conservative and nonconservative. Any force is called <u>a conservative force</u> if the force depends only on position and if work done by the force on a particle moving between any two positions depends only on the initial and final positions and so is independent of the path gone.

For example, we may prove that the force of gravity is a conservative force. We know that during the falling of mass m the work done by the gravitational force is $W_g = mgh$, where h is the vertical height through which an object falls.

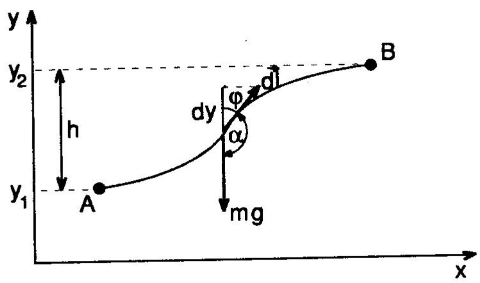

Figure 4 - 4

We now suppose an object moves along some arbitrary path in the xy plane, as shown in Fig. 4-4. By Eq. 4-2 we calculate the work done by gravity:

$$W_g = \int_A^B \vec{F}_g \, d\vec{\ell} = \int_A^B mg \cos\alpha \, d\ell \, . \qquad (4\text{-}9)$$

As $\varphi = 180^\circ - \alpha$, the angle φ between vector $d\vec{\ell}$ and its vertical component dy holds $\cos\varphi = -\cos\alpha$ and we see $dy = d\ell \cos\varphi = -d\ell \cos\alpha$.

Now we have

$$W_g = -\int_{y_1}^{y_2} mg \, dy = -mg(y_2 - y_1) \, . \qquad (4\text{-}10)$$

We see that the work done depends only on the vertical height $h = y_2 - y_1$ and does not depend on the path gone. So, we may say that gravity is a conservative force. Note that in our case in Fig. 4-4 $y_2 > y_1$ and therefore the work done by gravity is negative. If $y_2 < y_1$, the object is falling and the work done by gravity would be positive.

> There is also the other <u>equivalent definition</u> of a <u>conservative force</u>: a force is a conservative one if the work done by the force is zero whenever a particle moves along any closed path that returns it to its original position.

But not all forces are conservative. For example, the force of friction is a nonconservative force, since the work done by friction is equal to the product of the friction force and distance traveled. So, the work done by this force depends on the path length.

4 - 6 Potential Energy

In this section we introduce potential energy (PE) which is associated with the position of a body and which can be defined only in relation to a conservative force and which is (like KE) closely related to the concept of work.

Such example of potential energy is gravitational potential energy. We define the change in gravitational potential energy U when an object moves from a height y_1 to a height y_2 relative to any horizontal surface as

$$\Delta U = U_2 - U_1 = mg(y_2 - y_1) \,. \qquad (4\text{-}11)$$

This equation defines the change in potential energy between two points.

In section 4-5 we saw: if an object follows path as in Fig. 4-4, work done by gravity is (see Eq. 4-10)

$$W_g = - mg(y_2 - y_1) \,. \qquad (4\text{-}12)$$

By comparing Eqs. 4-12, 4-11 and 4-9, we see that the change in gravitational potential energy is equal to the negative of the work done by gravity when the object moves from the point A of height y_1 to the point B of height y_2 :

$$\Delta U = - U_g = - \int_A^B \vec{F}_g \, d\vec{\ell} \,. \qquad (4\text{-}13)$$

Besides gravitational there are other types of potential energy.

> In general we define the change in potential energy associated with a conservative force $\vec{F}$ as the negative of the work done by that force:

$$\Delta U = U_B - U_A = - \int_A^B \vec{F} \, d\vec{\ell} = - W \,. \qquad (4\text{-}14)$$

This definition makes sense only for conservative forces such as gravity, it does not apply to nonconservative forces like friction.

When we know the potential energy U as a function of coordinates x, y, z we may write the relation between the force F and the potential energy U(x, y, z) :

$$F(x, y, z) = - \left(\vec{i} \frac{\partial U}{\partial x} + \vec{j} \frac{\partial U}{\partial y} + \vec{k} \frac{\partial U}{\partial z} \right) = - \text{grad}\, U(x, y, z) \,. \qquad (4\text{-}15)$$

So, the components of $\vec{F}$ are

$$F_x = - \frac{\partial U}{\partial x} \,, \qquad F_y = - \frac{\partial U}{\partial y} \,, \qquad F_z = - \frac{\partial U}{\partial z} \,. \qquad (4\text{-}16)$$

4 - 7 Mechanical Energy and its Conservation

Let us consider a conservative system in which energy is transformed from kinetic to potential or vice versa. According to the work-energy theorem (Eq. 4-8) the net work W done on a particle is equal to the change in KE of the particle:

$$W = \Delta KE .$$

Since we assume a conservative system, the net work done can be written in terms of the changing potential energy (Eq. 4-14):

$$\Delta U = - W .$$

So, we can write

$$\Delta KE + \Delta U = 0 . \qquad (4\text{-}17)$$

We see, that if the kinetic energy of the system increases, the potential energy decreases by an equal amount and vice versa.

We now define a quantity E, called <u>total mechanical energy</u> of system, as the sum of the kinetic and potential energy:

$$E = KE + U ,$$

and from Eq. 4-17 we have for conservative forces only

$$E = KE + U = \text{constant}, \qquad (4\text{-}18)$$

that is, the <u>total mechanical energy of a conservative system remains constant</u>, or we say that the <u>total mechanical energy is conserved</u>. This is called the <u>principle of conservation of mechanical energy for conservative forces</u>.

For example, if v_1 and U_1 represent the velocity and potential energy at one instant, and v_2 , U_2 at a second instant, we can rewrite Eq. 4-18 in form

$$\frac{1}{2} mv_1^2 + U_1 = \frac{1}{2} mv_2^2 + U_2 . \qquad (4\text{-}19)$$

<u>E x a m p l e :</u> The simple pendulum of mass m is released at $t = 0$ when the cord makes an angle $\varphi = \varphi_o$ to the vertical (see Fig. 4-5; the length of a massless cord is L).

a) Determine the speed as a function of position and at the lowest point.
b) Determine the tension in the cord.
(Ignore friction and air resistance.)

<u>Solution</u>: a) Two forces are acting on the bob at any moment: gravity mg and the force F_c the cord exerts on the bob. The latter always acts perpendicular to the motion, so it does no work. The total mechanical energy of the system is

$$E = \frac{1}{2} mv^2 + mgy ,$$

where y is the vertical height of the bob at any moment. If we take $y = 0$ at the lowest point, hence at $t = 0$

$$y = y_o = L - L \cos \varphi_o = L(1 - \cos \varphi_o) ,$$

and at $t = 0$

$$E = mgy_o$$

since $v = 0$ at $t = 0$.

At any other moment

$$E = \frac{1}{2} mv^2 + mgy = mgy_o .$$

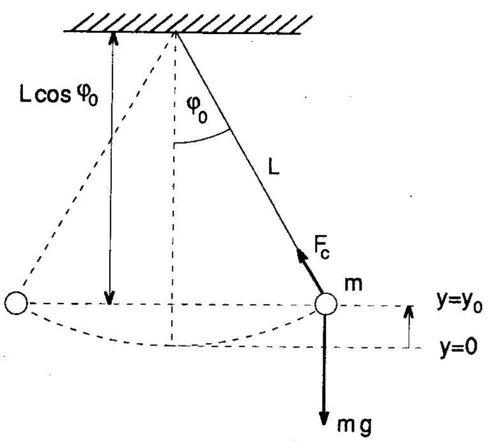

Figure 4 - 5

From this equation

$$v = \sqrt{2g(y_o - y)},$$

or

$$v = \sqrt{2gL(\cos\varphi - \cos\varphi_o)}.$$

At the lowest point, $y = 0$, $\varphi = 0$

$$v = \sqrt{2gy_o} \quad \text{or} \quad v = \sqrt{2gL(1 - \cos\varphi_o)}.$$

b) The tension in the cord is the force F_c that the cord exerts on the bob. There is no work done by this force since this force is perpendicular to the motion.

Radial acceleration of the bob is v^2/L; the net force in the radial direction is equal F_c minus component of gravity $mg\cos\varphi$ that acts outward. Hence

$$m\frac{v^2}{L} = F_c - mg\cos\varphi.$$

Thus

$$F_c = m\left(\frac{v^2}{L} + g\cos\varphi\right).$$

We use the result for v^2:

$$F_c = m\left[2g(\cos\varphi - \cos\varphi_o) + g\cos\varphi\right] = mg(3\cos\varphi - 2\cos\varphi_o).$$

4 - 8 Gravitational Potential Energy and Escape Velocity

So far we assumed the force of gravity $F = mg$ is constant. But this assumption is accurate for objects near the surface of the earth.

Generally, the gravitational force exerted by the earth on a particle of mass m decreases inversely as the square of the distance r from the earth's center. This relationship is given by Newton's law of universal gravitation (as we saw in Chapter 3):

$$\vec{F} = -G\frac{mM}{r^2}\vec{r}^{\,o},$$

where M is the mass of the earth, $\vec{r}^{\,o}$ is a unit vector directed radially from the earth's center. The minus sign indicates that the force on m is directed toward the earth's center, in the direction opposite to $\vec{r}^{\,o}$.

We now suppose an object of mass m moves from one position (point 1) to another (point 2) along an arbitrary path so that its distance from the earth's center changes from r_1 to r_2 (see Fig. 4-6).

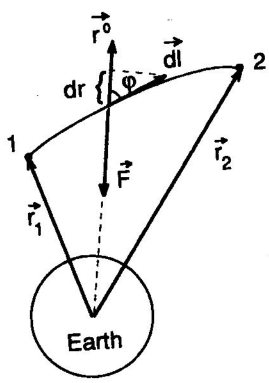

Figure 4 - 6

The work done by the gravitational force is

$$W = \int_1^2 \vec{F}\,d\vec{\ell} = -GmM\int_1^2 \frac{\vec{r}\,d\vec{\ell}}{r^2}, \qquad (4\text{-}20)$$

where $d\vec{\ell}$ represents an infinitesimal displacement. Since $\vec{r}^{\,o} d\vec{\ell} = dr$ is increment of $d\vec{\ell}$ along $\vec{r}^{\,o}$ (see Fig. 4-6), then

$$W = -GmM\int_{r_1}^{r_2}\frac{dr}{r^2} = GmM\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

Since the work of the gravitational force depends only on the position of the end points of path and not on the path, the gravitational force must be a conservative force, and we can determine the potential energy for the gravitational force. The change in potential energy is defined as the negative of the work done by the force (see section 4-6). So we have

$$\Delta U = U_2 - U_1 = -W = -\frac{GmM}{r_2} + \frac{GmM}{r_1} . \qquad (4\text{-}21)$$

So, the potential energy at any distance r from the earth's center can be expressed

$$U(r) = -\frac{GmM}{r} + C ,$$

where C is a constant. Usually we choose $C = 0$, so that

$$U(r) = -\frac{GmM}{r} . \qquad (4\text{-}22)$$

with this choice for C we have $U = 0$ at $r = \infty$. We see that potential energy any object is always negative. The total energy of a particle of mass m conserved, since gravity is a conservative force. So,

$$\frac{1}{2} mv_1^2 - G\frac{mM}{r_1} = \frac{1}{2} mv_2^2 - G\frac{mM}{r_2} = \text{constant.}$$

When a body is projected from the earth, it will return to earth unless its speed is very high. But if the speed is high enough, it will continue out into space never to return to earth. The minimum initial speed needed to prevent an object from returning to the earth is called the escape velocity. To calculate the value of the escape velocity, we must substitute $r_1 = R = 6{,}3.10^6$ m the radius of the earth, $r_2 = \infty$, $v_2 = 0$; we obtain

$$v_{esc} = \sqrt{\frac{2GM}{R}} = 1{,}12.10^4 \text{ m/s},$$

or 11,2 km/s.

4 - 9 Central Force

A central force is defined as any force whose magnitude depends only on the distance r from some single point, called the origin or the center, and whose direction is either toward or away from this origin.

Such a force can be express as

$$\vec{F} = F(r)\,\vec{r^o} . \qquad (4\text{-}23)$$

The work done by the central force is

$$W = \int_1^2 \vec{F}\,\vec{d\ell} = \int_1^2 F(r)\,\vec{r^o}\,\vec{d\ell} = \int_{r_1}^{r_2} F(r)\,dr , \qquad (4\text{-}24)$$

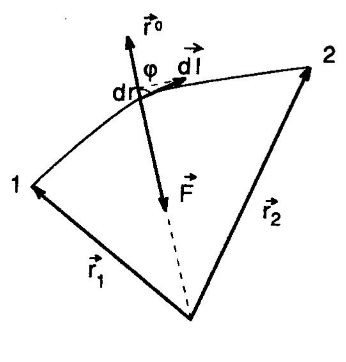

Figure 4 - 7

since $\vec{r^o}\,\vec{d\ell} = dr$.

We see that the work depends only on the end points r_1, r_2 of the path. So, any central force is a concervative force. The change of the potential energy of a central force is

$$\Delta U = U_2 - U_1 = -W = -\int_1^2 \vec{F}\,\vec{d\ell} = -\int_{r_1}^{r_2} F(r)\,dr , \qquad (4\text{-}25)$$

and we see that it is function only of r : $U = U(r)$.

4 - 10 Stable and Unstable Equilibrium

Let us assume the motion is in one dimension x and we are given the potential energy of a particle of mass m as a function of position U(x).

The total mechanical energy E must be constant:

$$\frac{1}{2} mv^2 + U(x) = E = \text{constant},$$

so that

$$v = \frac{dx}{dt} = \sqrt{\frac{2}{m}\left[E - U(x)\right]}. \qquad (4\text{-}26)$$

For example, let a graph of U(x) versus x be as follows:

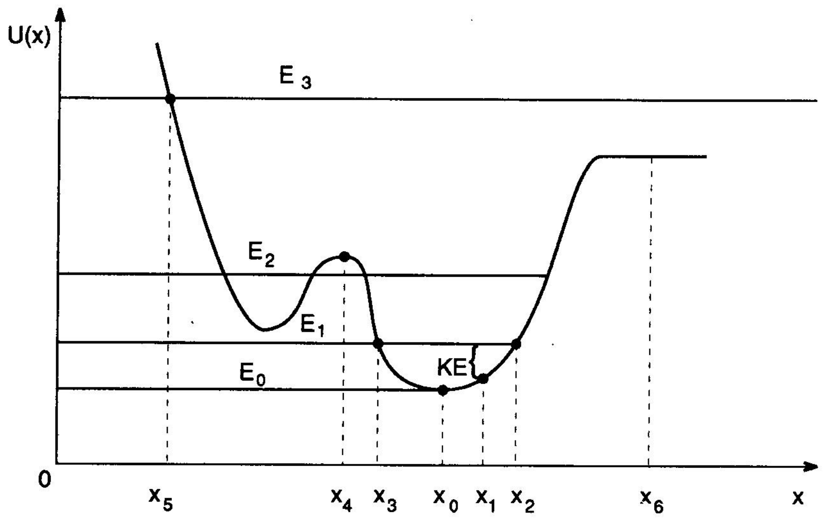

Figure 4 - 8

The total energy E is constant and thus can be represented as a horizontal line on this graph. What the actual value of E will be for a given system depends on the initial conditions. For example, the total energy E of a particle oscillating on the end of a spring depends on the amount the spring is initially compressed or stretched. On the Fig. 4-8 we have such four total energy lines of some system.

From Eq. 4-26 it is clear that U(x) must be less or equal E :

$$U(x) \leq E .$$

So, the minimum value which the total energy can have for the PE shown in Fig. 4-8 is that labeled E_0. For this value of E, the particle has only potential energy but no kinetic energy and thus for this value of E the particle can only be at rest at position $x = x_0$.

If the total energy E is greater than E_0, say it is E_1 on our plot, the particle can have both kinetic and potential energy and

$$KE = E - U(x) .$$

Our curve represents U(x) at each x. So, the kinetic energy for any x is represented by the distance between the E and the magnitude of U(x). In our graph the KE of a particle at x_1 when its total energy is E_1 is indicated by "KE". It is clear that a particle with total energy E_1 can oscillate only between the points x_2 and x_3, since for $x > x_2$ or $x < x_3$ the potential energy would be greater than E_1. At x_2 and x_3 the total energy E_1 equals the potential energy, thus the kinetic energy is zero and the velocity of a particle is also zero. These points are called the turning points of the motion.

If the particle has energy $E = E_2$, there are four turning points. The particle can now move in only one of the two potential "valleys", depending on where it is initially. It cannot get from one valley to the other - for example at x_4 $U(x) > E_2$. For energy $E = E_3$, there is only one turning point at $x = x_5$ since $U(x) < E_3$ for all $x > x_5$.

We know the force F is related to the potential energy U by

$$F = -\frac{dU}{dx},$$

that is, the force is equal to the negative of the slope of the U - versus - x curve at any point.

At $x = x_0$ the slope is zero, so F = 0. At such a point the particle is said to be in equilibrium, since the net force on the particle is zero. So, its acceleration is zero. When the particle displaced slightly from $x = x_0$ it returns back its equilibrium point and the particle is said to be in stable equilibrium. Any minimum in the potential energy curve represents a point of stable equilibrium.

At point $x = x_4$ it is also $F = -dU/dx = 0$ a particle would also be in equilibrium. But the particle will not return to equilibrium if displaced slightly, it moves away from the equilibrium point. Points like x_4, where the potential energy curve has a maximum, are points of unstable equilibrium.

When a particle is in region of constant potential energy, such as $x = x_6$ in Fig. 4-8, the force is zero for any x of this region. The particle is said to be in neutral equilibrium.

5. MANY BODIES MECHANICS

Up to now we have been mainly concerned with the motion of a single particle. When we have dealt with an object, that is, a body that has size, we have assumed that it underwent only translational motion, we have assumed that our body approximated an ideal particle. Real bodies, however, can undergo rotational motion as well. A basic idea for study of such bodies is that of center of mass. Later in this chapter we discuss linear momentum and its conservation.

5 - 1 Center of Mass

General motion of a real body (or system of bodies) can be considered as the sum of the translational motion of its center of mass (cm) plus rotational motion about its center of mass.