6. ROTATIONAL MOTION ABOUT AN AXIS

In this chapter we will deal with rotational motion of rigid bodies. By a rigid body we mean a body that has a definite shape which doesn´t change. Of course any real body is able of deforming when a force acts on it. But these effects are often small, so the concept of an ideal rigid body is a good approximation in many cases. By rotational motion we mean that all points in the body move in circles and that the centers of these circles lie on a line called the axis of rotation.

6 - 1 Kinematics of Rotational Motion

Every particle in a body rotating about a fixed axis moves in a circle whose center is on the axis and the radius of this circle is R, where R represents the perpendicular distance of the particle from the axis of rotation.

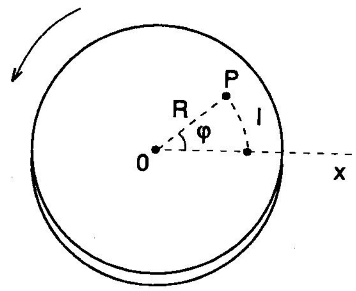

Figure 6 - 1

Perpendicular line drawn from the axis to any particle sweeps out the same angle φ in the same time. To indicate how far the body has rotated, we specify the angle φ of some particular line in the body with respect to some reference line, such as the x axis (see Fig. 6-1). A particle of the body (such as P in Fig. 6-1) travels the distance ℓ of its circular path.

In general, any angle φ is given by

$$\varphi = \frac{\ell}{R},$$

where R is the radius of the circle and ℓ is the arc length subtended by the angle φ, specified in radians.

In Chapter 1 we defined the angular velocity ω and angular acceleration ε, as

$$\omega = \frac{d\varphi}{dt} \quad \text{and} \quad \varepsilon = \frac{d\omega}{dt} .$$

ω and ε are now the same for all points in the body. Thus ω and ε are properties of the body as a whole.

By Eqs. 1-26, 1-27 and 1-28 the angular velocity ω and angular acceleration ε of a body can be related to the linear velocity v and to the acceleration a of any point in the body:

$$v = R\omega,$$
$$a_T = R\,\varepsilon,$$
$$a_R = \omega^2 R,$$

where R is the perpendicular distance of the point from the rotation axis, and a_R and a_T are the radial and tangential components of the linear acceleration. It is clear, v, a_R and a_T are different for particles at different distances R from the axis.

The frequency of rotation f of the body, measured in revolutions per second, is related to the angular velocity

$$\omega = 2\pi f .$$

From Chapter 1 we have known the equations for the velocity and the path traveled of the uniform linear acceleration motion. The same equations can be derived for <u>constant angular acceleration motion</u> if we replaced x by φ, v by ω and a by ε:

Linear	Angular	
$v = v_0 + at$	$\omega = \omega_0 + \varepsilon t$	
$x = v_0 t + \frac{1}{2} at^2$	$\varphi = \omega_0 t + \frac{1}{2} \varepsilon t^2$	(a, ε constant)

6 - 2 Vector Nature of Angular Quantities

The linear quantities displacement, velocity, and acceleration are vectors. We will see in this section that angular velocity and angular acceleration can also be treated as vectors, although angular displacement, φ is not a vector.

First let us see why the angular displacement cannot be a vector. One property of a vector is that when two vectors are added, you get the same result no matter in what order you add them. That is, if the two vectors are called $\vec{V}_1$ and $\vec{V}_2$, then

$$\vec{V}_1 + \vec{V}_2 = \vec{V}_2 + \vec{V}_1 .$$

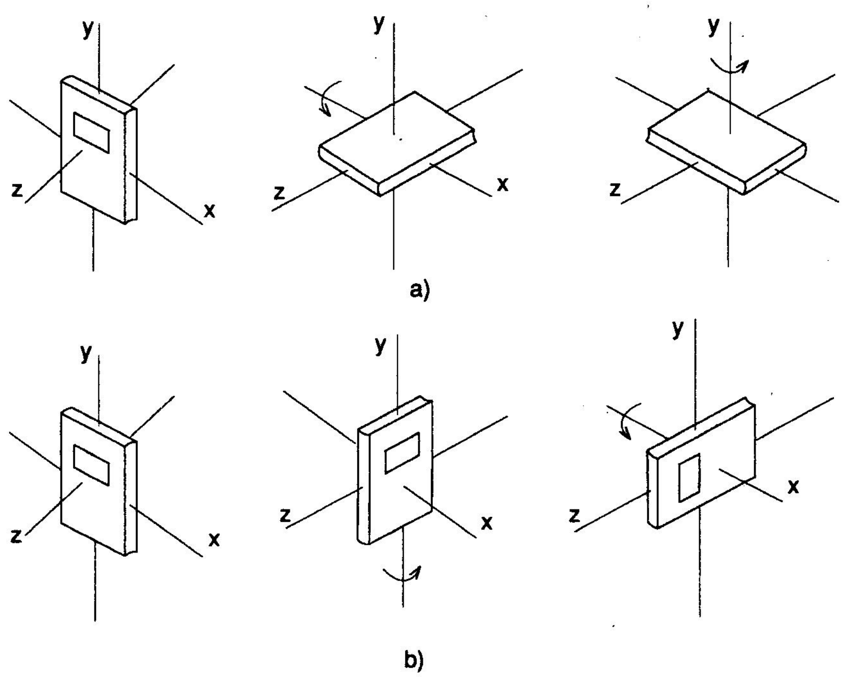

Figure 6 - 2

Suppose, now, we rotate a book by $\varphi_1 = 90^\circ$ around the x axis followed by a rotation $\varphi_2 = 90^\circ$ around the y axis, as shown in Fig. 6-2a. If, instead, we first rotate the book by $\varphi_2 = 90^\circ$ around the y axis followed by $\varphi_1 = 90^\circ$ around the x axis, Fig. 6-2b, we do not get the same result! In other words, $\varphi_1 + \varphi_2 \neq \varphi_2 + \varphi_1$. Hence φ cannot be a vector.

But now consider a rotation $\varphi_1 = 15^\circ$ about the x axis and $\varphi_2 = 15^\circ$ about the y axis. In this case, $\varphi_1 + \varphi_2$ is nearly (but not quite) equal to $\varphi_2 + \varphi_1$ as shown in Fig. 6-3. However, in the limit of infinitesimal angles of rotation, the equality $d\varphi_1 + d\varphi_2 = d\varphi_2 + d\varphi_1$ is exact. We get the same result when we add two infinitesimal rotation angles in either order. Hence an infinitesimal angular displacement, $d\varphi$, satisfies the commutative law and is a vector, although a finite angular displacement is not.

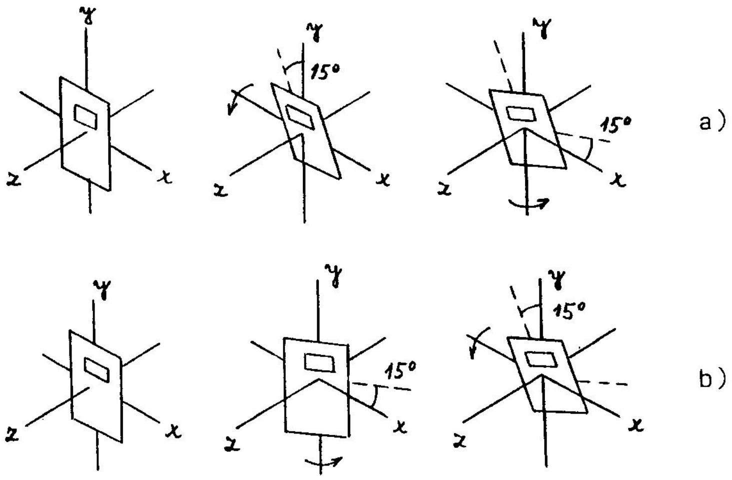

Figure 6-3

The angular velocity, ω , must also be a vector since it is the product of a vector $(d\vec{\varphi})$ and a scalar $(1/dt)$:

$$\vec{\omega} = \frac{d\vec{\varphi}}{dt} .$$

Similarly, since $\vec{\omega}$ is a vector, the angular velocity

$$\vec{\varepsilon} = \frac{d\vec{\omega}}{dt}$$

is also a vector.

We choose the axis of rotation to be the direction of the angular velocity vector $\vec{\omega}$. An orientation of vector $\vec{\omega}$ is given by right-hand rule: when the fingers of the right-hand are curled around the rotation axis and point in the direction of the rotation, then the thumb points in the orientation of $\vec{\omega}$. Note that no particle of the rotating body moves in the direction of $\vec{\omega}$.

If the axis of rotation is fixed, then $\vec{\omega}$ can change only in magnitude and thus vector $\vec{\varepsilon} = d\vec{\omega}/dt$ must also point along the axis of rotation. If the axis of rotation changes direction, vector $\vec{\omega}$ changes direction too, and in this case vector $\vec{\varepsilon}$ will not point along the axis of rotation.

6 - 3 Torque

In present section we will discuss the causes of rotational motion. Firstly, we define the lever arm $R_\perp$ as the perpendicular distance of the axis of rotation from the line of action of the force (we mean the distance which is perpendicular to both the axis of rotation and to an imaginary line drawn along the direction of the force). See Fig. 6-4a.

The product of the force times the lever arm is called the moment of the force about the axis or torque τ. So, we can write the torque about a given axis as

$$\tau = R_\perp F , \tag{6-1}$$

where $R_\perp$ is the lever arm (see Fig. 6-4a).

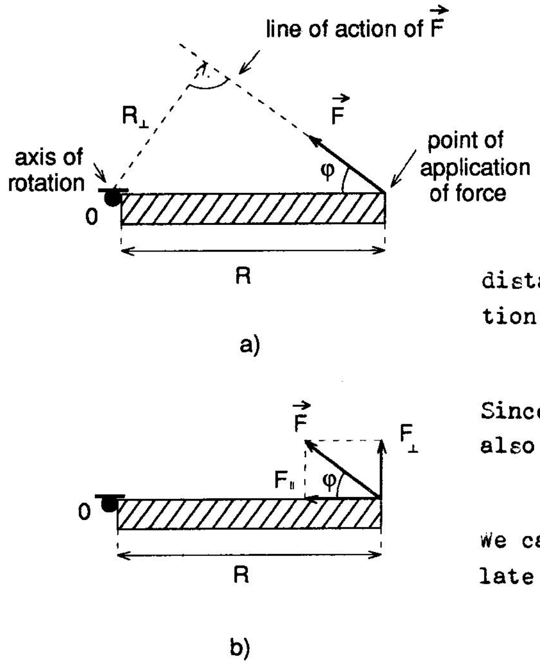

Figure 6 - 4

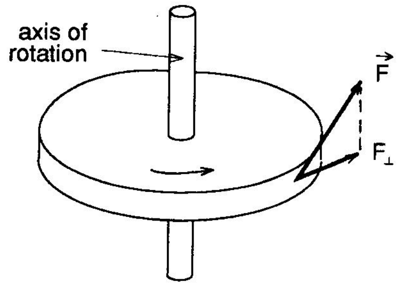

Figure 6 - 5

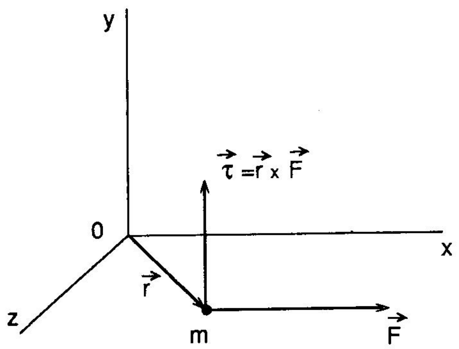

Figure 6 - 6

The second equivalent way of determining the torque of the force is to resolve the force into components parallel and perpendicular to a line joining the point of application of the force to the axis as in Fig. 6-4b. In this case the torque will be equal to $F_\perp$ times the distance R from the axis to the point of application of the force:

$$\tau = RF_\perp . \qquad (6\text{-}2)$$

Since $F_\perp = F \sin \varphi$ and $R_\perp = R \sin \varphi$, we have also for the torque

$$\tau = RF \sin \varphi . \qquad (6\text{-}3)$$

We can use any of Eqs. 6-1, 6-2 and 6-3 to calculate the torque.

The unit of the torque is N.m in SI units.

Notice that since we are interested only in rotation about a fixed axis in all this chapter, we consider only forces that act in a plane perpendicular to the axis of rotation.

If a force doesn't act in a plane like this, we must take its component into a plane perpendicular to the axis; this component can give rise to rotation about the axis (see Fig. 6-5).

We now express the torque as a vector. For a particle of mass m on which a force $\vec{F}$ is applied, the torque about a point 0 is

$$\vec{\tau} = \vec{r} \times \vec{F} , \qquad (6\text{-}4)$$

where $\vec{r}$ is the position vector of the particle relative to point 0 (Fig. 6-6). If we have a system of particles (which could be the particles making up a rigid body) the total torque $\vec{\tau}$ on the system is the sum of the torques on the individual particles:

$$\vec{\tau} = \sum \vec{r}_i \times \vec{F}_i ,$$

where $\vec{r}_i$ is the position vector of the i-th particle and $\vec{F}_i$ is the net force on the i-th particle.

6 - 4 Torque and Rotational Inertia

Firstly, let us consider a particle of mass m rotating in a circle of radius R at the end of a rod whose mass we can ignore (Fig. 6-7).

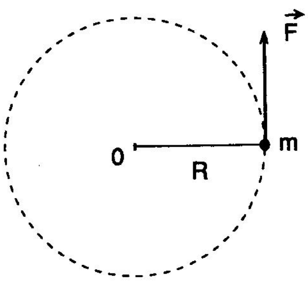

Figure 6 - 7

The torque which gives rise to its angular acceleration ε is $\tau = RF$. If we use the relation between the angular acceleration and the tangential acceleration $a_T = R\varepsilon$, we can write the second law of motion for our particle

$$F = ma_T = mR\varepsilon$$

and the torque is now given by

$$\tau = mR^2\varepsilon. \qquad (6\text{-}5)$$

Eq. 6-5 represents a relation between the angular acceleration and the applied torque τ for a single particle. The quantity mR^2 represents the <u>rotational inertia</u> of the particle and is called its <u>moment of inertia</u>.

Now let us consider a rotating rigid body. We may assume it to be consisting of many particles located at various distances from the axis of rotation. For the torque τ_i of the i-th particle of the body we may write

$$\tau_i = m_i R_i^2 \varepsilon , \qquad (6\text{-}6)$$

where m_i and R_i are the mass and distance from the axis of rotation of the i-th particle, respectively. ε is angular acceleration which is the same for all the particles of the body.

The sum of the various torques is the <u>total torque</u> of the body

$$\tau = \sum \tau_i = \varepsilon \sum m_i R_i^2 . \qquad (6\text{-}7)$$

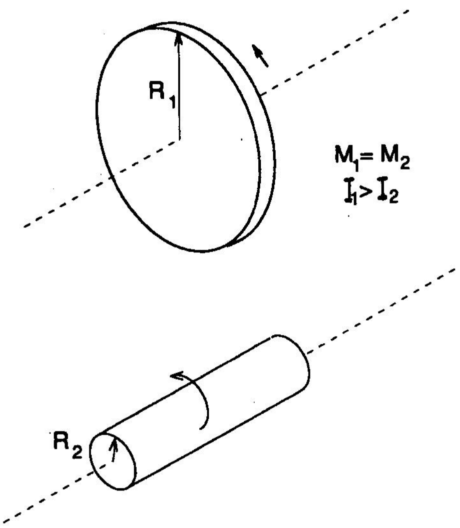

Figure 6 - 8

The sum $\sum m_i R_i^2 = m_1R_1^2 + m_2R_2^2 + \ldots + m_nR_n^2$ in Eq. 6-7 represents the sum of the masses of each particle in the body multipled by the squared perpendicular distance of each particle from the axis of rotation. This quantity is called the <u>rotational inertia</u> or <u>moment of inertia</u> I of the body:

$$I = \sum m_i R_i^2 . \qquad (6\text{-}8)$$

The moment of inertia I has unit of $kg.m^2$ in SI units.

From Eqs 6-7 and 6-8 we can write

$$\tau = I\varepsilon. \qquad (6\text{-}9)$$

This equation is valid for the rotation of a rigid body about a fixed axis.

We see that the moment of inertia I, which is a measure of the rotational inertia of a body, plays the same role for rotational motion that mass does for translational motion. From Eq. 6-8 can be seen that the rotational inertia of an object depends not only on its mass but also on how that mass is distributed. For example, a large-diameter cylinder will have greater rotational inertia than one of equal mass but smaller diameter (and therefore longer) see Fig. 6-8. The former will be harder to start rotating and harder to stop.

E x a m p l e 1 : A 15 N force (represented by F_T) is applied to a cord wrapped around a wheel of radius R_o = 33 cm, Fig. 6-9. The wheel is observed to accelerate uniformly from rest to reach an angular speed of 30 s^{-1} in 3 s. If there is a frictional torque τ_{fr} = 1.1 N.m, determine the moment of inertia of the wheel.

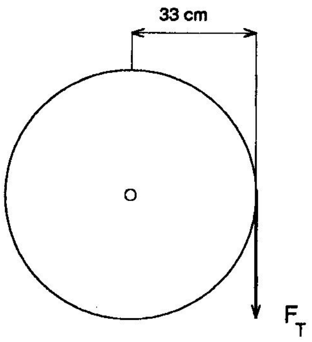

Figure 6 - 9

Solution: The net torque is the applied torque due to F_T minus the frictional torque:

$$\tau = F_T R_o - \tau_{fr} ,$$

$$\tau = 0.33 \times 15 - 1.1 = 3.85 \text{ N.m.}$$

The angular acceleration is

$$\varepsilon = \frac{\Delta\omega}{\Delta t} = \frac{30 - 0}{3} = 10 \text{ s}^{-2} .$$

Hence

$$I = \frac{\tau}{\varepsilon} = \frac{3.85 \text{ N.m}}{10 \text{ s}^{-2}} = 0.385 \text{ kg.m}^2 .$$

E x a m p l e 2 : Calculate the angular acceleration ε of the wheel and the linear acceleration a of the mass m . Determine also the angular velocity ω of the wheel and the linear velocity v of the mass m at time t if the wheel starts from rest at t = 0. Assume a frictional torque is τ_{fr}, the moment of inertia of the wheel is I (see Fig. 6-10).

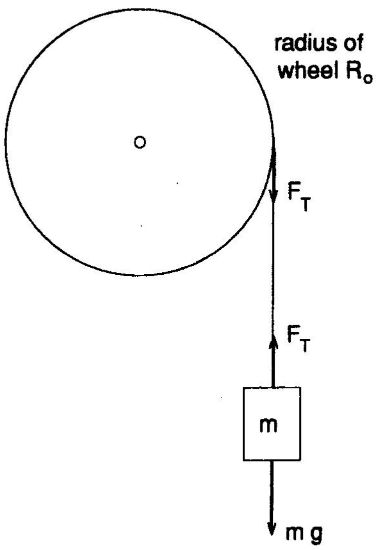

Figure 6 - 10

Solution: We can write for the rotation of the wheel:

$$\tau = I\varepsilon ,$$

where

$$\tau = F_T R_o - \tau_{fr} ,$$

so, the angular acceleration of the wheel is

$$\varepsilon = \frac{\tau}{I} = \frac{F_T R_o - \tau_{fr}}{I} .$$

Next we look at the linear motion of the mass m . Two forces act on the mass (see Fig. 6-10): the force of gravity mg acts downward and the tension of the cord F_T upward. So, we can write

$$ma = mg - F_T .$$

If we use the relation $a = R_o \varepsilon$, we have from last two equations

$$\varepsilon = \frac{(mg - m R_o \varepsilon) R_o - \tau_{fr}}{I} ,$$

or

$$\varepsilon = \frac{mg R_o - \tau_{fr}}{I + mR_o^2} .$$

We may see that all quantities on the right-hand side are constants, thus the angular acceleration of the rotation of the wheel is constant.

So, we can now express the linear acceleration a of the mass m, the angular velocity ω of the wheel and the linear velocity v of the mass:

$$a = R_o \varepsilon ,$$

$$\omega = \omega_o + \varepsilon t = \varepsilon t , \qquad \text{(since } \omega_o = 0 \text{ at } t = 0)$$

$$v = R_o \omega = R_o \varepsilon t .$$

Example 3: A uniform rod of mass M and length L can pivot freely about a hinge as shown in Fig. 6-11. The rod is released from horizontal position. At the moment of release, determine the angular acceleration of the rod. Assume the force of gravity acts at the center of mass of the rod.

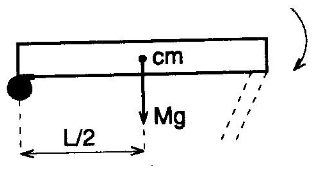

Figure 6 - 11

Solution: The only torque on the rod is

$$\tau = Mg \frac{L}{2} .$$

The moment of inertia of a uniform rod pivoted about its end is

$$I = \frac{1}{3} ML^2 .$$

Thus

$$\varepsilon = \frac{\tau}{I} = \frac{3}{2}\frac{g}{L} .$$

That is the angular acceleration at the moment of release. As the rod descends, the torque cannot be constant and thus the rod's angular acceleration also cannot be constant.

6 - 5 Calculation of Moment of Inertia

Many bodies can be considered as a continuous distribution of mass. For many bodies or systems of particles the moment of inertia can be calculated directly. In this case Eq. 6-8 defining moment of inertia has form

$$I = \int R^2 dm , \qquad (6\text{-}10)$$

where dm represents the mass of any infinitesimal element of the body and R is the perpendicular distance of this element from the axis of rotation. The integral is taken over the whole body mass.

Example 1: Calculate the moment of inertia of a uniform hollow cylinder of inner radius R_1, outer radius R_2 and mass M, if the rotation axis is along the axis of symmetry.

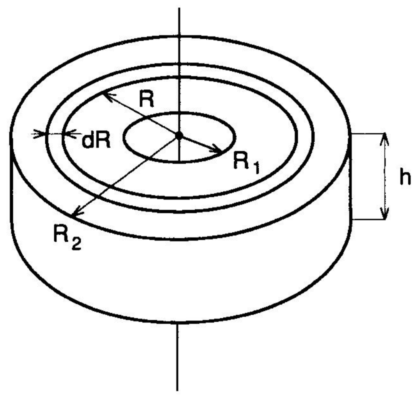

Figure 6 - 12

Solution: We divide the cylinder into concentric cylindrical rings of thickness dR. Its mass is

$$dm = \rho \, dV ,$$

where ρ is density and dV is its volume

$$dV = 2\pi R \, dR \, h .$$

So,

$$dm = 2\pi \rho h R \, dR .$$

The moment of inertia is obtained by integrating

$$I = \int R^2 \, dm = 2\pi \varrho h \int_{R_1}^{R_2} R^3 \, dR = \frac{1}{2} \pi \varrho h (R_2^4 - R_1^4) \; .$$

The volume of hollow cylinder is

$$V = (\pi R_2^2 - \pi R_1^2) h$$

and its mass

$$M = \varrho V = \varrho \pi (R_2^2 - R_1^2) h \; .$$

Thus

$$I = \frac{\pi \varrho h}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2) = \frac{1}{2} M (R_1^2 + R_2^2) \; .$$

Note that for a solid cylinder $R_1 = 0$ and if we put $R_2 = R_o$, we have for it

$$I = \frac{1}{2} M R_o^2 \; .$$

Example 2 : Calculate the moment of inertia of a uniform solid sphere of radius r_o and mass M about an axis through its center.

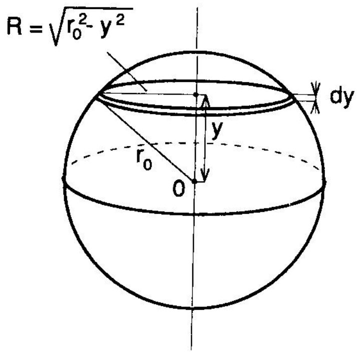

Figure 6 - 13

Solution: We divide the sphere into infinitesimal cylinders of thickness dy. Each cylinder has a radius

$$R = \sqrt{r_o^2 - y^2}$$

and a mass

$$dm = \varrho \, dV = \varrho \pi R^2 \, dy = \varrho \pi (r_o^2 - y^2) \, dy \; .$$

Hence, the moment of inertia of each infinitesimal cylinder is

$$dI = \frac{1}{2} dm \, R^2 = \frac{\pi \varrho}{2} (r_o^2 - y^2)^2 \, dy = \frac{\pi \varrho}{2} (r_o^4 - 2 r_o^2 y^2 + y^4) \, dy \; .$$

By integrating over all these infinitesimal cylinders

$$I = \int dI = \frac{\varrho \pi}{2} \int_{-r_o}^{r_o} (r_o^4 - 2 r_o^2 y^2 + y^4) \, dy = \frac{8}{15} \pi \varrho r_o^5 \; .$$

Since the volume of a sphere is

$$V = \frac{4}{3} \pi r_o^3$$

and its mass

$$M = \varrho V = \frac{4}{3} \pi \varrho r_o^3 \, ,$$

so

$$I = \frac{2}{5} M r_o^2 \; .$$

6 - 6 Parallel - Axis Theorem

Parallel-axis theorem states that if I is the moment of inertia of a body of total mass M about any axis, and I_{cm} is the moment of inertia about an axis passing through the center of mass and parallel to the first axis at distance h away, then

$$I = I_{cm} + Mh^2 . \tag{6-11}$$

The proof of this theorem is as follows:

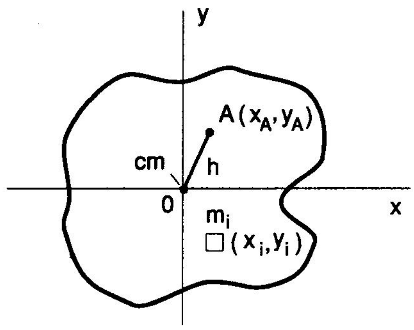

Figure 6 - 14

Let us choose our coordinate system so the origin is at the cm. Let I_{cm} be the moment of inertia about z axis. Let I represent the moment of inertia of the body about an axis parallel to the z axis that passes through the point A which has coordinates x_A and y_A (see Fig. 6-14).

So the moment of inertia I about the axis through point A is

$$I = \sum m_i \left[(x_i - x_A)^2 + (y_i - y_A)^2\right], \tag{6-12}$$

where x_i, y_i and m_i represent the coordinates and mass of an arbitrary point in the body and thus the expression

$$\left[(x_i - x_A)^2 + (y_i - y_A)^2\right]$$

is the square of the distance from this point to the point A. The equation 6-12 can be rewritten in form

$$I = \sum m_i (x_i^2 + y_i)^2 - 2x_A \sum m_i x_i - 2y_A \sum m_i y_i + (x_A^2 + y_A^2) \sum m_i .$$

The first term on the right is just $I_{cm} = \sum m_i (x_i^2 + y_i^2)$ since the cm is at the origin. The second and third terms are zero since, by definition of the cm, $\sum m_i x_i = \sum m_i y_i = 0$ because $x_{cm} = y_{cm} = 0$. The last term is Mh^2 since $\sum m_i = M$ and $(x_A^2 + y_A^2) = h^2$ where h is the distance of A from the cm. Thus we have proved $I = I_{cm} + Mh^2$.

6 - 7 Angular Momentum and its Conservation

The equation

$$\tau = I \varepsilon$$

describes the rotation of a rigid body about a fixed axis. This relation is the rotational equivalent of the second law of motion for translational motion

$$F = ma = \frac{d(mv)}{dt} = \frac{dp}{dt},$$

where $p = mv$ is the linear momentum.

An analogous relation can be written for the rotational motion of a rigid body: since the angular acceleration $\varepsilon = d\omega/dt$, we have

$$\tau = I \varepsilon = I \frac{d\omega}{dt} = \frac{d(I\omega)}{dt} = \frac{dL}{dt}, \tag{6-13}$$

where the quantity $L = I\omega$ is called the <u>angular momentum</u> of the body about its axis of rotation.

From Eq. 6-13 it is clear that if the net torque τ acting on the rotating body is zero, then

$$\frac{dL}{dt} = 0 \quad \text{and} \quad L = I\omega = \text{constant}$$

This is the <u>law of conservation of angular momentum</u> for a rotating body:

> the total angular momentum of a rotating body remains constant if the net torque acting on it is zero.

So, when there is zero net torque acting on a body, we can write

$$I\omega = I_0\omega_0 = \text{constant},$$

where I_0 and ω_0 are the moment of inertia and angular velocity at some initial time ($t = 0$), and I and ω are their values at some other time.

We now express angular momentum as a vector.

Suppose a particle of mass m has momentum $\vec{p}$ and position vector $\vec{r}$ with respect to the origin O in some chosen reference frame. Then the angular momentum $\vec{\ell}$ of the particle about point O is defined as the vector cross product of $\vec{r}$ and $\vec{p}$:

$$\vec{\ell} = \vec{r} \times \vec{p} \quad [\text{particle}] . \qquad (6\text{-}14)$$

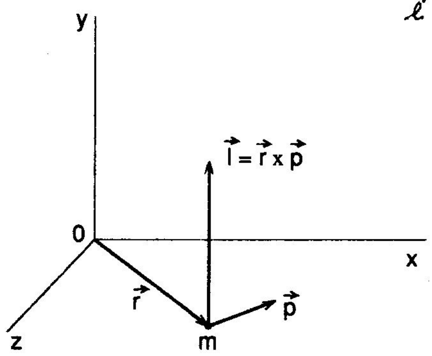

Figure 6 - 15

Its direction is perpendicular to both $\vec{r}$ and $\vec{p}$ as given by the right-hand rule (Fig. 6-15). Its magnitude is given by

$$\ell = rp \sin\theta$$

or

$$\ell = rp_{\perp} = r_{\perp}p ,$$

where θ is the angle between $\vec{r}$ and $\vec{p}$ and $p_{\perp}$ ($= p \sin\theta$) and $r_{\perp}$ ($= r \sin\theta$) are the components of $\vec{p}$ and $\vec{r}$ perpendicular to $\vec{r}$ and $\vec{p}$ respectively.

Now let us find the relation between angular momentum and torque for a particle. If we take the derivative of $\vec{\ell}$ with respect to time we have

$$\frac{d\vec{\ell}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} .$$

But

$$\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0 ,$$

since $\sin\theta = 0$ for this case. Thus

$$\frac{d\vec{\ell}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} .$$

If we let $\vec{F}$ represent the resultant force on the particle, then in reference frame, $\vec{F} = d\vec{p}/dt$ and

$$\vec{r} \times \vec{F} = \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d\vec{\ell}}{dt} .$$

But $\vec{r} \times \vec{F} = \vec{\tau}$ is the net torque on our particle. Hence

$$\vec{\tau} = \frac{d\vec{\ell}}{dt} . \qquad (6\text{-}15)$$

The time rate of change of angular momentum of a particle is equal to the net torque applied to it.

E x a m p l e : Determine the angular momentum of a particle of mass m moving with speed v in a circle of radius r in a counterclockwise direction.

Solution: The value of the angular momentum depends on the choice of the point O.

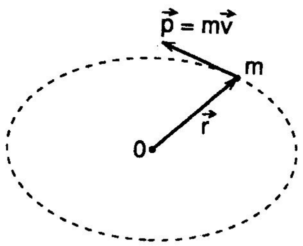

Figure 6 - 16

We calculate $\vec{\ell}$ with respect to the center of the circle (Fig. 6-16). Then $\vec{r}$ is perpendicular to $\vec{p}$ so $\ell = |\vec{r} \times \vec{p}| = rmv$. Since $v = \omega r$ and $I = mr^2$ for a single particle rotating about an axis a distance r away, we can write

$$\ell = mvr = mr^2\omega = I\omega .$$

The direction of $\vec{\ell}$ is perpendicular to the plane of the circle.

6 - 8 Relation Between Torque and Angular Momentum Vectors

Let $\vec{r}_i$ be the position vector of the i-th particle in a reference frame, and $\vec{r}_{cm}$ be the position vector of the center of mass of the system in this reference frame. The position of the i-th particle with respect to the cm is $\vec{r}_i^*$ where (see Fig. 6-17)

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i^* .$$

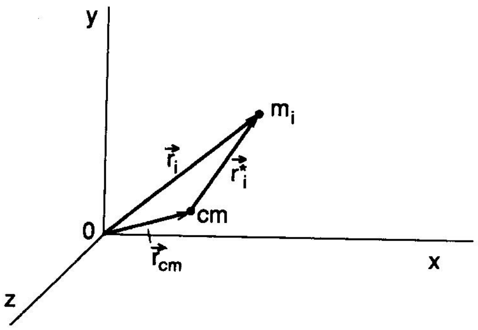

Figure 6 - 17

We take the derivative of this equation:

$$\vec{p}_i = m_i \frac{d\vec{r}_i}{dt} = m_i \frac{d}{dt} (\vec{r}_i^* + \vec{r}_{cm}) = \vec{p}_i^* + m_i \vec{v}_{cm} . \quad (6\text{-}16)$$

The angular momentum of the system with respect to the cm is

$$\vec{L}_{cm} = \sum (\vec{r}_i^* \times \vec{p}_i^*) .$$

We take the time derivative

$$\frac{d\vec{L}_{cm}}{dt} = \sum\left(\frac{d\vec{r}_i^*}{dt} \times \vec{p}_i^*\right) + \sum\left(\vec{r}_i^* \times \frac{d\vec{p}_i^*}{dt}\right).$$

The first term equals zero since $\vec{v}_i^* \times m\vec{v}_i^* = 0$, so (by using Eq. 6-16)

$$\frac{d\vec{L}_{cm}}{dt} = \sum \vec{r}_i^* \times \frac{d}{dt} (\vec{p}_i - m_i \vec{v}_{cm}) = \sum \vec{r}_i^* \times \frac{d\vec{p}_i}{dt} - \left(\sum m_i \vec{r}_i^*\right) \times \frac{d\vec{v}_{cm}}{dt} .$$

The second term is equal zero, since $\sum m_i \vec{r}_i^* = M \vec{r}_{cm}^*$, but $\vec{r}_{cm}^* = 0$ by definition (the position of the cm is at origin of the cm reference frame).

We use

$$\frac{d\vec{p}_i}{dt} = \vec{F}_i ,$$

where $\vec{F}_i$ is the net force on m_i.

Then

$$\frac{d\vec{L}_{cm}}{dt} = \sum \vec{r}_i^* \times \vec{F}_i = \sum (\vec{\tau}_i)_{cm} = \vec{\tau}_{cm} ,$$

where $\vec{\tau}_{cm}$ is the resultant torque on the system calculated about the cm.

We can summarize: the relation

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad (6\text{-}17)$$

is not valid in general. It is true only when $\vec{\tau}$ and $\vec{L}$ are calculated with respect to either (1) the origin of an inertial reference frame or (2) the center of mass of a system of particles or of a rigid body.

6 - 9 Rotational Kinetic Energy

We consider the rotating body. If R_i represents the perpendicular distance of any one particle of the body from the axis of rotation, then its linear velocity is $v_i = R_i\omega$. The total kinetic energy of the whole body will be the sum of the KE of all its particles:

$$KE = \sum \frac{1}{2} m_i v_i^2 = \frac{1}{2}\omega^2 \sum m_i R_i^2 = \frac{1}{2} I \omega^2 ,$$

where the angular velocity ω is the same for every particle, and I is the moment of inertia.

So, the kinetic energy of an object rotating about a fixed axis is

$$\text{rotational KE} = \frac{1}{2} I \omega^2 . \qquad (6\text{-}18)$$

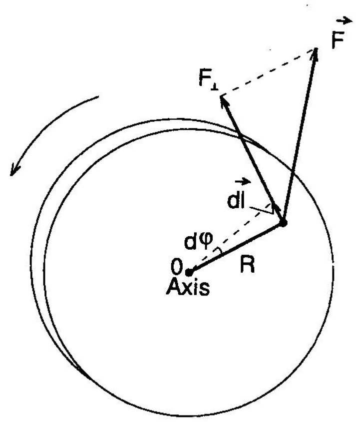

Figure 6 - 18

We now calculate the work done on a body rotating about a fixed axis. We suppose a force $\vec{F}$ exerted at a point a perpendicular distance R from the axis of rotation (see Fig. 6-18). The work done by this force is

$$W = \int \vec{F}\, d\vec{\ell} = \int F_{\perp} R\, d\varphi .$$

But $F_{\perp} R$ is the torque about the axis, so

$$W = \int \tau\, d\varphi . \qquad (6\text{-}19)$$

The work-energy theorem holds also for rotation of a rigid body about a fixed axis:
as we can write

$$\tau = I\varepsilon = I\frac{d\omega}{dt} = I\frac{d\omega}{d\varphi}\frac{d\varphi}{dt} = I\omega\frac{d\omega}{d\varphi} ,$$

then

$$W = \int_{\varphi_1}^{\varphi_2} \tau\, d\varphi = \int_{\omega_1}^{\omega_2} I\omega\, d\omega = \frac{1}{2} I\omega_2^2 - \frac{1}{2} I\omega_1^2 .$$

The work-energy theorem for a body rotating about a fixed axis states that the work done in rotating a body through an angle $\varphi_2 - \varphi_1$ is equal to the change in rotational kinetic energy of the body.

By Eq. 6-19, power P as the rate of work done, is

$$P = \frac{dW}{dt} = \tau\frac{d\varphi}{dt} = \tau\omega . \qquad (6\text{-}20)$$

Example: A rod of mass M is pivoted on a frictionless hinge at one end as shown in Fig. 6-19. The rod is held at rest horizontally and then released. Determine the angular velocity of the rod when it reaches the vertical position, and the speed of the rod's tip at this moment.

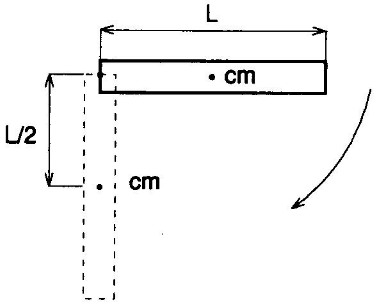

Figure 6 - 19

Solution: We can use the work-energy theorem here; the work done is due to gravity. The work done by gravity is, of course, equal to the change in gravitational potential energy of the rod. Since the cm of the rod drops a vertical distance L/2, the work done by gravity is

$$W = Mg \frac{L}{2} .$$

The initial KE is zero. Hence, from the work--energy theorem,

$$\frac{1}{2} I \omega^2 = Mg \frac{L}{2} .$$

Since $I = \frac{1}{3} ML^2$ for a rod pivoted about its end, we can solve for ω :

$$\omega = \sqrt{\frac{3g}{L}} .$$

The tip of the rod will have a linear speed

$$v = L\omega = \sqrt{3gL} .$$

7. EQUILIBRIUM AND ELASTICITY

7 - 1 Center of Gravity

We consider any body as made up of many particles, each of mass m_i. Although gravity acts on each of these particles, we can show that the sum of all these individual gravitational forces has the equivalent effect of a single force which acts at a single point called the center of gravity (cg). This force is equal to Mg, where $M = \sum m_i$ is the total mass of the body and g is acceleration due to gravity. If g has the same value at all parts of the body (which is the usual case), the position of the center of gravity is the same as that of the center of mass.

The total force of gravity on a body made up of n particles of masses $m_1, m_2, \ldots, m_n$ is

$$\vec{F} = m_1 \vec{g} + m_2 \vec{g} + \ldots + m_n \vec{g} = \sum m_i \vec{g} = M \vec{g} . \qquad (7\text{-}1)$$

So, a single force $\vec{F} = M\vec{g}$ will have the same effect on the translational motion of the body as does the sum of all the gravitational forces acting on the particles of the body.

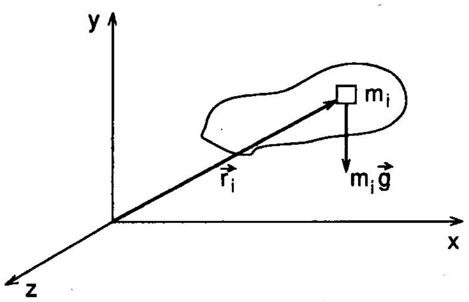

Figure 7 - 1

The position of the force $\vec{F}$ is given by condition so that the rotational motion of the body is the same as does the sum of all the forces of gravity acting on the particles. To determine this, we calculate the sum of all the torques on the body about some arbitrary point O, as shown in Fig. 7-1.

If $\vec{r}_i$ is the position vector of the i-th particle relative to O, then the sum of all the torques due to gravity acting on the particles of the body is