pressure on the wall than if there were no attractive forces. The gas acts as though it is subject to a pressure in excess of the externally applied pressure. This excess pressure can be expressed as

$$p_{excess} = \frac{n^2 a}{V^2}, \qquad (13-4)$$

where n is number of moles, a is a proportionality constant and V is the volume of the gas. In this way we can express pressure of the gas as

$$p' = p + \frac{n^2 a}{V^2}. \qquad (13-5)$$

Substituting Eq. (13-5) and (13-3) into ideal gas law - Eq. (10-23) we obtain

$$\left(p + \frac{n^2 a}{V^2}\right)(V - nb) = nRT. \qquad (13-6)$$

Equation (13-6) is known as van der Waals equation of state. The constants a and b are different for different gases and are determined experimentally.

14. CHARGE AND FIELD

The science of electricity has its roots in the observation, known to Thales of Miletus in 600 B.C., that a rubbed piece of amber will attract bits of straw. The word "electricity" comes from the Greek word electron, which means "amber".

The word electricity may also evoke an image of modern technology, radio, television, microwave radar, motors etc. But the electric force plays even deeper role in our lives, since according to atomic theory, the forces that act between atoms and molecules to hold them together to form liquids and solids are electrical forces.

In this chapter we will discuss the development of ideas about electricity.

14 - 1 Electric Charge - Coulomb's Law

We can show that there are two kinds of charge by rubbing a glass rod with silk and hanging it from a long thread as it is seen in Fig. 14-1.

If a second rod is rubbed with silk and held near the rubbed end of the first rod, the rods will repel each other. On the other hand, a rod of plastic (sealing wax) rubbed with fur will attract the glass rod. Two plastic rods rubbed with fur will repel each other. We explain these facts by saying that rubbing a rod gives it an electric charge and that the charges on the two rods exert forces on each other. Clearly the charges on the glass and on the plastic must be different in nature.

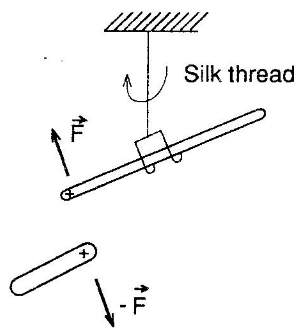

Figure 14 - 1

B. Franklin (1706 - 1790) named the kind of electricity that appears on the glass positive and the kind that appears on the plastic negative. These names have remained to this day.

We can sum up these experiments by saying that <u>like charges repel and unlike charges attract.</u>

When a glass rod is rubbed with silk, a positive charge appears on the rod. Measurement shows that a negative charge of equal magnitude appears on the silk. This suggests that rubbing does not create charge but merely transfers it from one object to another. The charge is separated but the sum of the two is zero. This is an example of the law of conservation of electric charge, which states that

the net amount of electric charge produced in any process is zero.

This conservation law is as firmly established as those for energy and momentum.

Only within the past century has it become clear that electric charge has its origin within the atom itself. Today's view shows the atom as having a heavy, positively charged nucleus surrounded by one or more negatively charged electrons. In its normal state, the positive and negative charges within the atom are equal, and the atom is electrically neutral.

The charge on one electron because of its fundamental nature is given the symbol e and is often referred to as the elementary charge:

$$e \approx 1{,}602 \quad 10^{-19}\ \mathrm{C}.$$

Note that e is defined as a positive number, so the charge on the electron is - e. The charge on a proton , which is a charged part of the nucleus, on the other hand is + e.

The SI unit of charge is the coulomb (abbr. C). A coulomb is defined as the amount of charge that flows through any cross section of a wire in 1 second if there is a steady current of 1 ampere in the wire.

We have already seen that an electric charge exerts a force on the other charge. Ch. A. Coulomb (1736 - 1806) measured electrical attractions and repulsions quantitavely and deduced the law that governs them. He concluded, the force one tiny charged object in vacuum(ideally a point charge) exerts on a second one is proportional to the product of the amount of charge on one, Q_1, times the amount of charge on the other Q_2 and inversely proportional to the square of the distance r between them; that is

$$F = k\frac{Q_1 Q_2}{r^2}, \qquad (14\text{-}1)$$

where k is a proportionality constant which is usually written in terms of another constant ε_0 called the permittivity of free space. It is related to k by

$$k = \frac{1}{4\pi\varepsilon_0}.$$

Coulomb's law can be then written

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}, \qquad (14\text{-}2)$$

where

$$\varepsilon_0 = 8{,}85418 \times 10^{-12}\ \mathrm{F.m^{-1}}.$$

Equation (14-2) gives the magnitude of the force that either object exerts on the other. The direction of this force is along the line joining the two objects. Notice that the force one charge exerts on the second is equal but opposite to that exerted by the second on the first. This is in accord with Newton's third law.

We can write Coulomb's law in vector form as

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q_1 Q_2}{r_{21}^2} \cdot \vec{r}^{\,o}_{21} , \qquad (14\text{-}3)$$

where $\vec{F}_{12}$ is the vector force on charge Q_1 due to Q_2 and $\vec{r}^{\,o}_{21}$ is the unit vector pointing from Q_2 toward Q_1. The charges Q_1 and Q_2 can be either positive or negative and this will affect the direction of the electric force. If Q_1 and Q_2 have the same sign, the product $Q_1Q_2 > 0$ and the force on Q_1 points away from Q_2 - that is, it is repulsive. If Q_1 and Q_2 have opposite signs, $Q_1Q_2 < 0$ and F_{12} points toward Q_2 - that is, it is attractive.

It should be also recognized that Coulomb's law applies to point charges and to charged objects whose size is much smaller than the distance between them. If several (or many) charges are present, the net force on any one of them will be the vector sum of the forces due to each of the others. For continuous distribution of charge this sum becomes an integral.

14 - 2 The Electric Field $\vec{E}$, Field Lines

According to M. Faraday an electric field extends outward from every charge and permeates all of space. When a second charge is placed near the first charge, it feels a force because of the electric field that is there. The electric field at the location of the second charge is considered to interact directly with this charge to produce the force.

To define the electric field operationally, we place a small test charge q_o (assumed positive for convenience) at the point in space that is to be examined, and we measure the electrical force $\vec{F}$ that acts on this charge. The electric field $\vec{E}$ at the point is defined as

$$\vec{E} = \frac{\vec{F}}{q_o} . \qquad (14\text{-}4)$$

Here $\vec{E}$ is a vector because $\vec{F}$ is one, q_o being a scalar. The direction of $\vec{E}$ is the direction of $\vec{F}$, that is, it is the direction in which a resting positive charge placed at the point would tend to move.

The SI unite of electric field $\vec{E}$ can be obtained from Eq. (14-4), that is

$$[E] = \frac{[F]}{[q_o]} = \frac{N}{C} = \frac{V}{m} .$$

The magnitude of the electric field is the force per unit positive charge. The commonly used SI unit of electric field $\vec{E}$ is $\frac{V}{m}$, where V is the abbreviation of Volt.

In order to visualize the electric field, we draw a series of lines to indicate the direction of the electric field at various points in space. These lines are called <u>electric field lines</u> or lines of force.

We summarize the properties of lines of force as follows:

1. The tangent to a line of force at any point gives the direction of $\vec{E}$ at that point.
2. The lines of force are drawn so that the number of lines per unit cross-sectional area (perpendicular to the lines) is proportional to the magnitude of $\vec{E}$. Where the lines are close together E is large and where they are far apart E is small.

3. The lines of force start only on positive charges and end only on negative charges.

Figure 14-2 shows the lines of force for two equal like charges and Figure 14-3 shows the lines of force for equal but opposite charges.

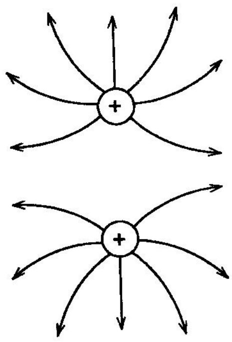

Figure 14 - 2

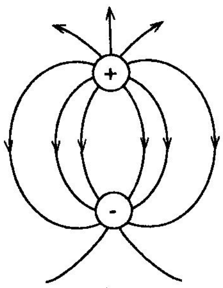

Figure 14 - 3

Let us now discuss the charge-field interaction by showing how we may calculate $\vec{E}$ for various points near given charge distributions, starting with the simple case of a point charge Q.

Let a test charge q_o be placed a distance r from a point charge Q. The magnitude of the force acting on q_o is given by Coulomb's law, or

$$F = \frac{1}{4\pi\varepsilon_o} \cdot \frac{q_o Q}{r^2} .$$

The electric field at the site of the test charge is given by Eq. (14-4), or

$$E = \frac{F}{q_o} = \frac{1}{4\pi\varepsilon_o} \cdot \frac{Q}{r^2} . \qquad (14\text{-}5)$$

The direction of $\vec{E}$ is on a radial line from Q, pointing outward if Q is positive and inward if Q is negative (see Fig. 14-4).

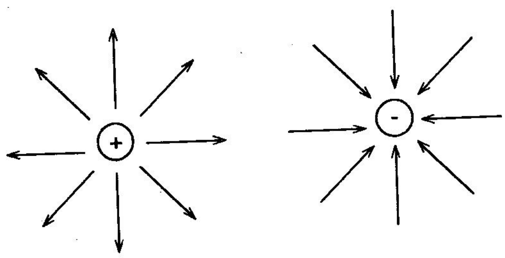

Figure 14 - 4

To find $\vec{E}$ for a group of point charges:

a) Calculate $\vec{E}_n$ due to each charge at the given point as if it were the only charge present.

b) Add these separately calculated fields vectorially to find the resultant field $\vec{E}$ at the point in equation form

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots = \Sigma \vec{E}_n \qquad n = 1, 2, 3, \ldots \qquad (14\text{-}6)$$

The sum is a vector sum, taken over all charges. Equation (14-6) is an example of the principle of superposition which states, in this context, that at a given point the electric fields due to separate charge distributions add up vectorially or superimpose independently. The principle of superposition for the electric field derives from experiment, and no exceptions have been observed.

If the charge distribution is a continuous one, the field it sets up at any point P can be computed by dividing the charge into infinitesimal elements dq. The field $d\vec{E}$ due to each element at the point in question is then calculated, treating the elements as point charges. The magnitude of $d\vec{E}$ (see Eq. 14-5) is given by

$$dE = \frac{1}{4\pi\varepsilon_o} \cdot \frac{dq}{r^2} , \qquad (14\text{-}7)$$

where r is the distance from the charge element dq to the point P. The resultant field at P is then found from the superposition principle by integrating the field contributions due to all the charge elements, or

$$\vec{E} = \int d\vec{E} . \qquad (14\text{-}8)$$

The integration is of course a vector operation.

Now let us discuss how to express the element of charge for the case when the charge distribution is continuous.

If the charge is distributed over a long wire it is useful to define the linear charge density, that is the charge per unit length as

$$\tau = \lim_{\Delta \ell \to 0} \frac{\Delta q}{\Delta \ell} = \frac{dq}{d\ell} \quad \left[\frac{C}{m}\right] \qquad (14\text{-}9)$$

The element of charge is then $dq = \tau \, d\ell$.

If the charge is distributed over a plane then the charge per unit area can be expressed as

$$\sigma = \lim_{S \to 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS} \quad \left[\frac{C}{m^2}\right] \qquad (14\text{-}10)$$

where σ is called the surface charge density.
The element of charge is then $dq = \sigma . dS$.

If the charge is distributed over a certain volume it is useful to define the volume charge density as

$$\rho_V = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \quad \left[\frac{C}{m^3}\right] \qquad (14\text{-}11)$$

The element of charge is then $dq = \rho_V \, dV$.

In this section we saw how we could use Coulomb's law to calculate $\vec{E}$ at various points if we knew enough about the distribution of charges that set up the field. This method always works, it is straightforward but, except in the simplest cases, laborious.

However as we shall see in Section 14-5, there exists another more simple method for calculation of electric field $\vec{E}$.

14 - 3 Motion of a Charged Particle in an Electric Field

In the preceding section we have seen how to determine $\vec{E}$ for some particular situations. Now let us suppose we know $\vec{E}$ and we want to find the force on a charged particle and its subsequent motion, that is let us determine the equation of the trajectory of particle.

Suppose a charged particle Q entering the uniform electric field at $x_o = y_o = z_o = 0$ with velocity $\vec{v}_o$ (see Fig. 14-5). The electric field $\vec{E}$ is pointed vertically upward. We can write the initial conditions as

$$t = 0 \qquad x_o = y_o = z_o = 0$$
$$\vec{v}_o(v_{ox}, v_{oy}, 0)$$

where $v_{ox} = v_o \cos\alpha$ and $v_{oy} = v_o \sin\alpha$.
Electric field $\vec{E}$ has the components $\vec{E}(0; E; 0)$.

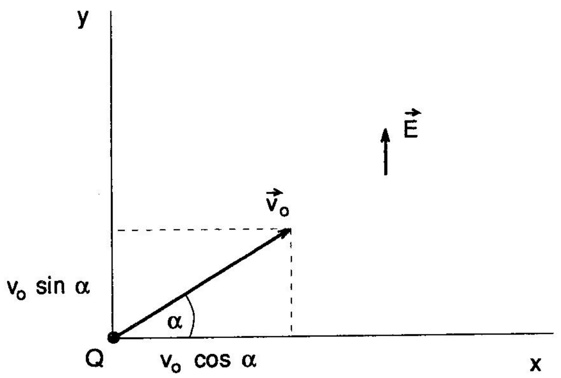

Figure 14 - 5

Because of initial conditions and the direction of the electric field vector $\vec{E}$ particle moves in a plane x - y.

From Eq. (14-4) we can obtain the expression for a force which exerts the electric field $\vec{E}$ on an object of charge Q as

$$\vec{F} = Q.\vec{E} . \tag{14-12}$$

From the Newton's second law we have

$$Q.\vec{E} = m \frac{d\vec{v}}{dt} . \tag{14-13}$$

Integrating Eq. (14-13) we obtain

$$\vec{v} = \frac{Q}{m} \vec{E}.t + \vec{v}_o . \tag{14-14}$$

Let us write the single vector Eq. (14-14) as the two scalar equations, that is

$$v_x = \frac{Q}{m} E_x t + v_{ox} ,$$

$$v_y = \frac{Q}{m} E_y t + v_{oy} .$$

With respect to initial conditions we obtain

$$v_x = v_o \cos \alpha ,$$

$$v_y = \frac{Q}{m} E.t + v_o \sin \alpha .$$

Integrating Eq. (14-14) we obtain for the position vector

$$\vec{r} = \frac{Q}{2m} \vec{E}\, t^2 + \vec{v}_o t + \vec{r}_o \tag{14-15}$$

We can write the single vector Eq. (14-15) as the two scalar equations,that is

$$x = \frac{Q}{2m} E_x t^2 + v_{ox} t + x_o$$

$$y = \frac{Q}{2m} E_y t^2 + v_{oy} t + y_o$$

With respect to initial conditions we have

$$x = v_o t \cos \alpha ,$$

$$y = \frac{Q}{2m} Et^2 + v_o t \sin \alpha .$$

Eliminating t yields

$$y = \frac{QE}{2mv_o^2 \cos^2\alpha} x^2 + x \operatorname{tg} \alpha \tag{14-16}$$

for the equation of the trajectory of the charged particle in an electric field. It is obvious that if an angle α which makes the initial velocity $\vec{v}_o$ with the positive x-direction equals to zero we can rewrite Eq. (14-16) as

$$y = \frac{QE}{2mv_o^2} x^2 , \tag{14-17}$$

which is the equation of parabola.

Note that in calculating the motion of a particle in a field set up by external charges the field due to the particle itself (that is, its self-field) is ignored. The gravity is also ignored.

The influence of electric field on the motion of charged particles is used for example in a cathode-ray oscilloscope for deflection of electron beam.

14 - 4 A Dipole in an Electric Field

In this section we shall study the electric field $\vec{E}$ of the positive and negative charge of equal magnitude Q placed a distance 2a apart as well as the influence of external electric field on this configuration of electric charges which is refered

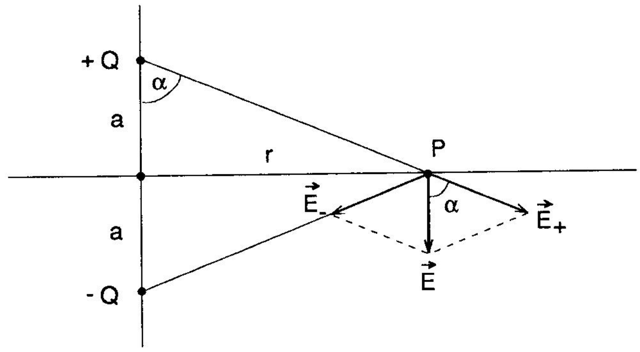

Figure 14-6

to as an electric dipole. The pattern of lines of force of electric dipole is shown in Fig. 14-3.

As we can seen in Fig. 14-6 we want to determine the field $\vec{E}$ due to the positive + Q and negative - Q charge at point P, a distance r along the perpendicular bisector of the line joining the charges. Assume $r \gg a$.

Using principle of superposition (see Eq. 14-6) we obtain for the total field at P

$$\vec{E} = \vec{E}_+ + \vec{E}_- ,$$

where $\vec{E}_+$ and $\vec{E}_-$ are the fields due to the positive and negative charges respectively. The magnitudes $\vec{E}_+$ and $\vec{E}_-$ are equal:

$$E_+ = E_- = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{(a^2 + r^2)} .$$

The horizontal component cancel at point P, so the vector sum of $\vec{E}_-$ and $\vec{E}_+$ points vertically downward and has the magnitude

$$E = 2E_+ \cos\alpha .$$

From the figure we see that

$$\cos\alpha = \frac{a}{\sqrt{a^2 + r^2}} .$$

Substituting the expressions for E_+ and $\cos\alpha$ into that for E yields

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{2aQ}{(a^2 + r^2)^{3/2}} .$$

If $r \gg a$ we can neglect a in the denominator; this equation then reduces to

$$E \approx \frac{1}{4\pi\varepsilon_0} \cdot \frac{2aQ}{r^3} . \qquad (14\text{-}18)$$

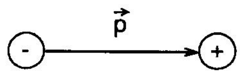

Figure 14-7

The product (2aQ) is called the electric dipole moment and is represented by the symbol p. The dipole moment can be considered to be vector of magnitude (2aQ), that points from the negative to the positive charge as shown in Fig. 14-7.

$$p = 2aQ . \qquad (14\text{-}19)$$

Thus we can rewrite Eq. (14-18) for distant points along the prependicular bisector, as

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{p}{r^3} . \qquad (14\text{-}20)$$

So the field decreases more rapidly for a dipole than for a single point charge ($1/r^3$ versus $1/r^2$) which we expect since at large distances the two opposite charges appear so close together as to neutralize each other. The $1/r^3$ dependence applies also for points not on the perpendicular bisector. In the second part of this section let us consider a dipole placed in a uniform external electric field $\vec{E}$ as shown in Fig. 14-8.

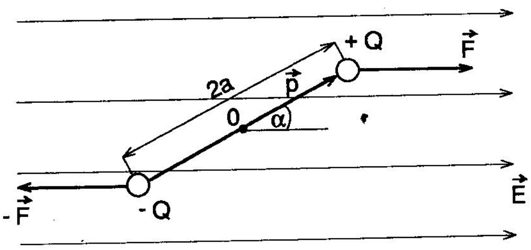

Figure 14-8

The dipole moment $\vec{p}$ makes an oriented angle α with the external electric field $\vec{E}$. Two equal and opposite forces $\vec{F}$ and $-\vec{F}$ act as shown, where

$$\vec{F} = Q.\vec{E} .$$

The net force is clearly zero. There will, however, be a torque on the dipole which has magnitude about the axis through 0 given by

$$\tau = QE.a \sin\alpha + (-QE)a \sin(\pi+\alpha) = (2a.Q)E \sin\alpha .$$

Recalling that $p = 2a.Q$, we obtain

$$\tau = pE \sin\alpha . \qquad (14\text{-}21)$$

We can write this equation in an vector notation

$$\vec{\tau} = \vec{p} \times \vec{E} . \qquad (14\text{-}22)$$

Thus an electric dipole placed in an external electric field $\vec{E}$ experiences a torque. The effect of the torque is to try to turn the dipole so $\vec{p}$ is parallel to $\vec{E}$.

If the electric field is not uniform, the force on the + Q of the dipole may not have the same magnitude as the force on the - Q, so there may be a net force as well as a torque.

Work (positive or negative) must be done by an external agent to change the orientation of an electric dipole in an external field. This work is stored as potential energy U in the system. If α in Fig. 14-8 has the initial value α_0, the work required to turn the dipole axis to an angle α is given as

$$U = W = \int dW = \int_{\alpha_0}^{\alpha} \tau \, d\alpha ,$$

where τ is the torque exerted by the agent that does the work. Combining this equation with Eq. (14-21) yields

$$U = \int_{\alpha_0}^{\alpha} pE \sin\alpha \, d\alpha = pE(-\cos\alpha)\Big|_{\alpha_0}^{\alpha} .$$

Since we are interested only in changes in potential energy, we can choose the reference orientation α_0 to have any convenient value, in this case 90^o. This gives

$$U = -pE \cos\alpha ,$$

or in vector symbolism

$$U = -\vec{p}.\vec{E} . \qquad (14\text{-}23)$$

Many molecules, such as a diatomic molecule CO, have a dipole moment (C has a small positive charge and O a small negative charge) even though the molecule as a whole is neutral, there is a separation of charge that results from an uneven sharing of electrons by the two atoms.

The electric dipoles play an important role for the understanding of dielectric properties of matter.

14 - 5 Electric Flux - Gauss's Law

Gauss's law, which we shall discuss in this section is a statement of the relation between electric charge and electric field. It is a more general and elegant form of Coulomb's law.

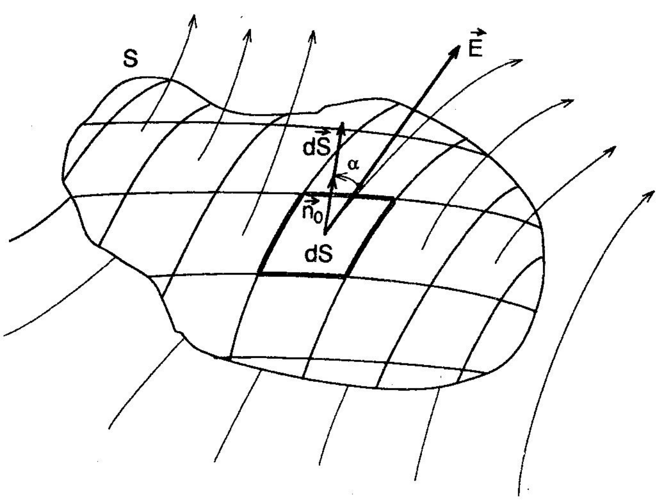

Figure 14 - 9

Before discussing Gauss's law itself we first discuss the concept of flux. Flux (symbol ϕ) is a property of all vector fields. We are concerned in this section with the flux of the electric field $\vec{E}$.

To define electric flux ϕ_E, consider Fig. 14-9 which shows an arbitrary surface S immersed in a nonuniform electric field $\vec{E}$. Let the surface be divided into elementary surfaces dS, each of which is small enough so that it may be considered to be plane. Since this elementary surface is infinitesimally small, $\vec{E}$ may be taken as a constant for all points in this surface.

Electric flux $d\phi_E$ through this elementary surface is defined as

$$d\phi_E = E\, dS_0 = E\, dS \cos\alpha , \qquad (14\text{-}24)$$

where dS_0 expresses the projection of dS on a surface perpendicular to $\vec{E}$ and α is the oriented angle between $\vec{E}$ and $\vec{dS}$. The area dS of a elementary surface can be represented by a vector $\vec{dS}$ that is

$$\vec{dS} = \vec{n}\, dS , \qquad (14\text{-}25)$$

where $\vec{n}$ is unit vector situated perpendicular to the surface dS. In fact, until now, there is an ambiguity in direction of the vector $\vec{dS}$. Thus for example in Fig. 14-9 the vector $\vec{dS}$ could point upward or downward. However Gauss's law deals with the flux through a closed surface - that is a surface that completely encloses a certain volume (like a sphere). For a closed surface, we define the direction of dS to point outward from the enclosed volume, see Fig. 14-10.

For a line entering the volume (on the left in Fig. 14-10), the angle α between $\vec{E}$ and $\vec{dS}$ must be greater than $\frac{\pi}{2}$, so $\cos\alpha < 0$. Hence, flux entering the enclosed volume is negative.

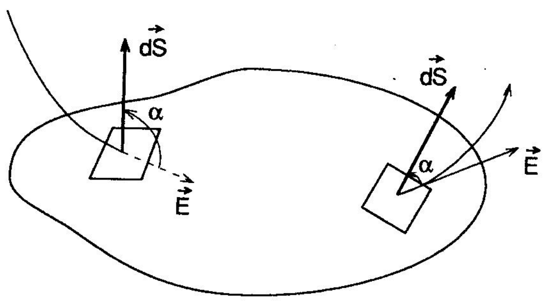

Figure 14 - 10

For a line leaving the enclosed volume (on the right in Fig. 14-10) the angle α must be less than $\frac{\pi}{2}$, so $\cos\alpha > 0$. Hence flux leaving the enclosed volume is positive.

Taking into account the above mentioned facts then with respect to Eq. (14-25) we can write Eq. (14-24) in vector notation as

$$d\phi_E = E(n\, dS)\cos\alpha = \vec{E}.d\vec{S} , \qquad (14\text{-}26)$$

where $d\phi_E$ expresses flux through the elementary surface dS. The net flux through the closed surface is then given by

$$\phi_E = \oint\!\!\oint_S \vec{E}.d\vec{S} , \qquad (14\text{-}27)$$

where the integration is over all an enclosing surface.

The flux through a surface is proportional to the number of electric field lines through it. If the number of lines that enter the volume is equal to the number of lines that leave then there is no net flux out of this surface. The flux, $\oint\!\!\oint_S \vec{E}.d\vec{S}$, will be nonzero only if some lines start or end within the volume. Since electric field lines start and stop only on electric charges, the flux will be nonzero only if the surface S encloses a net charge.

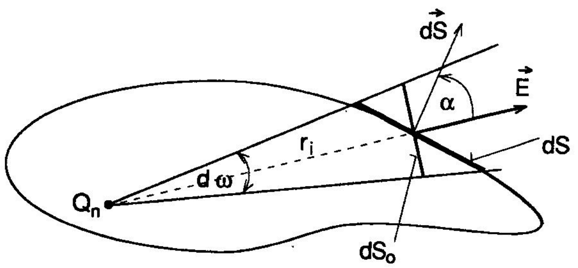

Figure 14 - 11

Let us therefore imagine that the surface S encloses electric charges $Q_1, Q_2, \ldots, Q_n$, see Fig. 14-11. Using principle of superposition (see Eq. 14-6) we can write Eq. (14-27) as

$$\oint\!\!\oint_S \vec{E}.d\vec{S} = \oint\!\!\oint_S \sum_{i=1}^{n} \vec{E}_i . d\vec{S} =$$

$$= \oint\!\!\oint \sum E_i \, dS_o .$$

Taking into account

$$dS_o = r_i^2 \, d\omega ,$$

where $d\omega$ is the solid angle, and Eq. (14-5) we can write

$$\oint\!\!\oint \sum_{i=1}^{n} E_i \, dS_o = \int_0^{4\pi} \sum_{i=1}^{n} \frac{Q_i}{4\pi\varepsilon_o r_i^2} r_i^2 \, d\omega = \sum_{i=1}^{n} Q_i \frac{1}{4\pi\varepsilon_o} \int_0^{4\pi} d\omega = \frac{\sum_{i=1}^{n} Q_i}{\varepsilon_o}$$

Thus we have obtained Gauss's law as

$$\oint\!\!\oint_S \vec{E}.d\vec{S} = \frac{\sum_{i=1}^{n} Q_i}{\varepsilon_o} \qquad (14\text{-}28)$$

that states that

electric flux through a closed surface equals to the net charge enclosed by that surface divided by a permittivity of free space.

Note that $\sum Q_i$ is the net charge, taking its algebraic sign into account. It doesn't matter where or how the charge is distributed within the surface. Charge outside the closed surface makes no contribution to the electric flux $\oint\!\!\oint_S \vec{E}.d\vec{S}$.

Let us prove this statement in the following way. Imagine a hypothetical closed cylinder of radius R immersed in a uniform electric field $\vec{E}$ for example in a vicinity of a charged plane, see Fig. 14-12.

We shall determine flux ϕ_E through this closed surface. This flux can be written as the sum of three terms, an integral over the left cylinder cap S_1, the cylindrical surface S_p and the right cap S_2. Thus

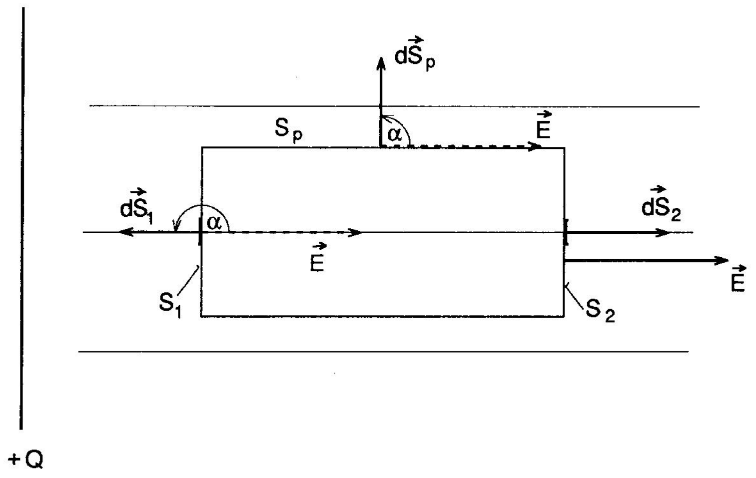

Figure 14 - 12

$$\phi_E = \oint_S \vec{E}.\vec{dS} = \oint_{S_1} \vec{E}.\vec{dS_1} + \oint_{S_p} \vec{E}.\vec{dS_p} + \oint_{S_2} \vec{E}.\vec{dS_2} .$$

For the left cap, the angle α equals 180^o, $\vec{E}$ has a constant value and the vectors $\vec{dS_1}$ are paralel. Thus

$$\iint_{S_1} \vec{E}.\vec{dS_1} = \iint_{S_1} E \cos 180^o \, dS_1 = - ES_1 .$$

Similary for the right cap (α = 0)

$$\iint_{S_2} \vec{E}.\vec{dS_2} = \iint_{S_2} E \cos 0 \, dS_2 = ES_2 .$$

Finally, for the cylinder wall

$$\iint_{S_p} \vec{E}.\vec{dS_p} = 0$$

because $\alpha = 90^o$ for all points on the cylindrical surface. Thus

$$\phi_E = - ES + 0 + ES = 0 .$$

We see that the charge situated outside the closed surface makes no contribution to the electric flux ϕ_E.

We can conclude this section with the statement that the Gauss´s law tells us that any difference between the input and output electric flux over any closed surface is due to charge within that surface.

14 - 6 Applications of Gauss´s Law

Gauss´s law offers a simple way to determine the electric field when the charge distribution is simple and symmetrical. In order to do this, however, we must choose the surface (for the integral on the left side of Gauss´s law - this surface is sometimes called "Gaussian" surface) very carefully so we can determine $\vec{E}$. We normally try to think of a surface that has just the symmetry needed so $\vec{E}$ will be constant on it or on its parts.

E x a m p l e 1 : Solid sphere of charge.

An electric charge $+Q$ is distributed uniformly throughout a nonconducting sphere of radius R, see Fig. 14-13. Determine the electric field E outside the sphere $(r > R)$ and inside the sphere $(r < R)$.

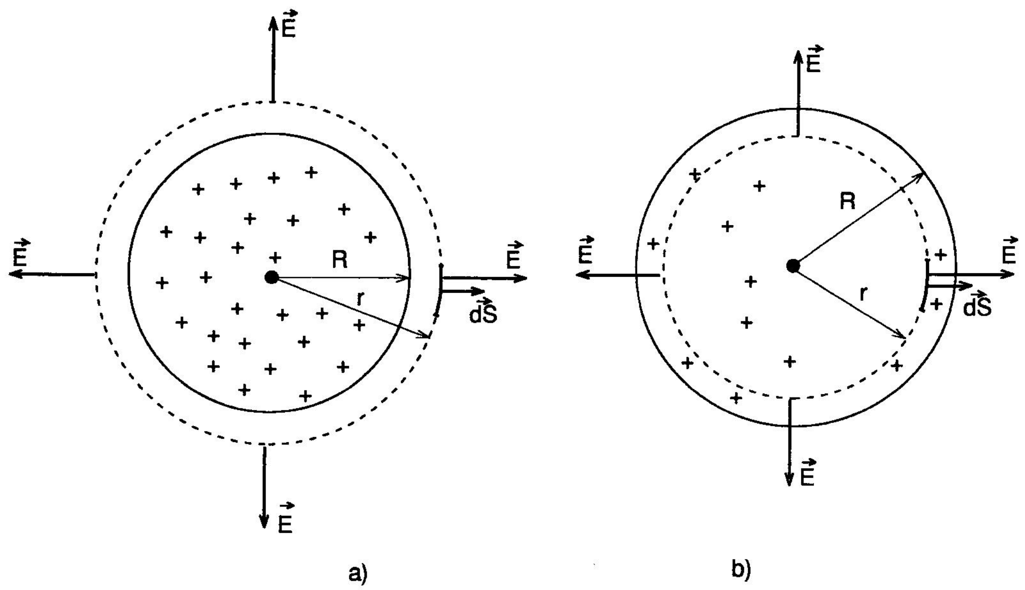

Figure 14-13

Solution: The volume charge density ρ (see Eq. 14-11) has a constant value for all points within a sphere. Since the charge is distributed symmetrically the electric field at all points of the sphere must be also symmetric that is $\vec{E}$ is directed radially outward. The vector $\vec{E}$ is therefore perpendicular to the surface so that the angle between $\vec{E}$ and $d\vec{S}$ equals to zero.

First let us determine the electric field outside the sphere, that is for $r > R$. For Gaussian surface we choose a sphere of radius $r > R$ see Fig. 14-13a. Thus we have

$$\oint\!\!\oint_S \vec{E}.d\vec{S} = \oint\!\!\oint_S E \cos 0^\circ \, dS = E\, 4\pi r^2 = \frac{Q}{\varepsilon_o},$$

or

$$E = \frac{Q}{4\pi\varepsilon_o r^2}. \qquad (14\text{-}29)$$

Thus for points outside the charged nonconducting sphere, the electric field has the value that it would have if the charge were concentrated at its center.

Inside the sphere we choose for Gaussian surface a concentric sphere of radius $r < R$, see Fig. 14-13b. Gauss's law gives

$$\oint\!\!\oint_S \vec{E}.d\vec{S} = \frac{Q'}{\varepsilon}$$

in which Q' is that part of Q which is contained within the sphere of radius r. For a uniform charge distribution we can write

$$Q = \rho_V \frac{4}{3}\pi R^3, \qquad Q' = \rho_V \frac{4}{3}\pi r^3,$$

or

$$Q' = Q\frac{r^3}{R^3}.$$

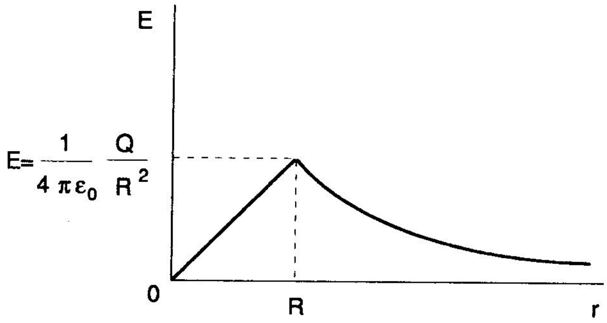

Figure 14 - 14

For Gauss's law we obtain

$$\oint_S \vec{E}.d\vec{S} = E\,4\pi r^2 = \frac{Q}{\varepsilon_o}\,\frac{r^3}{R^3},$$

or

$$E = \frac{Q}{4\pi\varepsilon_o R^3}\,r \qquad (14\text{-}30)$$

Magnitude of the electric field as a function of the distance r from the center of a uniformly charged nonconducting sphere is shown in Fig. 14-14.

E x a m p l e 2 : An infinite sheet of charge.

A positive electric charge is distributed uniformly, with a surface charge density σ over a very large nonconducting sheet. What is $\vec{E}$ at a distance r in front of the sheet?

Solution: We choose as a Gaussian surface a small closed cylinder of cross-sectional area S and height 2r, arranged to pierce the plane as shown in Fig. 14-15. Because of the symmetry we expect $\vec{E}$ to be directed perpendicular to the sheet of charge on both sides as shown, and to be uniform over the end caps of the cylinder. Since E does not pierce the cylindrical surface, there is no contribution to the flux from this source.

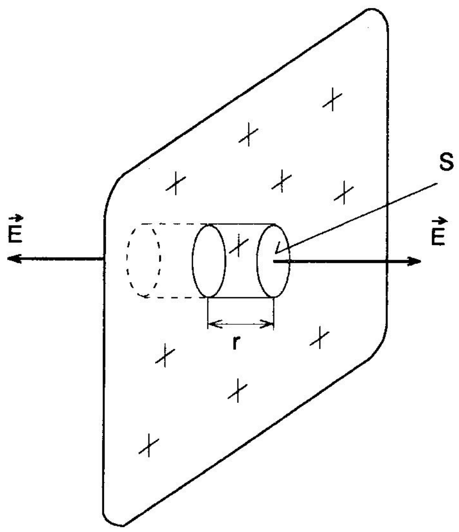

Figure 14 - 15

Thus Gauss's law gives

$$\oint_S \vec{E}.d\vec{S} = (ES + ES) = \frac{\sigma S}{\varepsilon_o},$$

where $(\sigma.S)$ is the enclosed charge.

The electric field is then

$$E = \frac{\sigma}{2\,\varepsilon_o}. \qquad (14\text{-}31)$$

Note that E is the same for all points on each side of the sheet. The field is uniform for points far from the ends of the plane, and close to its surface.

Let us now imagine that an electric charge is distributed uniformly over two parallel plates. The first one is charged positively, the second one is charged negatively with a surface charge density σ, see Fig. 14-16. We shall determine the electric field $\vec{E}$ between and outside these plates.

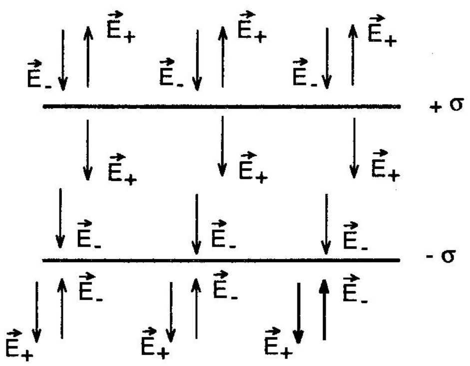

Figure 14 - 16

To determine the electric field $\vec{E}$ between the plates it is posible to use the principle of superposition (see Eq. 14-6).

We can see that between the plates is

$$\vec{E} = \vec{E}_+ + \vec{E}_-$$

where E_+ and E_- are the electric fields caused by positively and negatively charged plates respectively. With respect to Eq. (14-31) we obtain

$$E = \frac{\sigma}{\varepsilon_0} . \qquad (14\text{-}32)$$

From Fig. 14-16 we can see that the electric field outside the plates equals to zero.

14 - 7 Electric Field and Conductors

In the static situation that is, when the charges are at rest, the electric field inside any conductor must be zero. Otherwise, the free charges in the conductor would move, until the net force on each, and hence $\vec{E}$, were zero.

The direction of $\vec{E}$ for points close to the surface is at right angles to the surface, pointing away from the surface if the charge is positive. If $\vec{E}$ were not normal to the surface, it would have a component lying in the surface. Such a component would act on the charge carriers in the conductor and set up surface currents Since there are no such currents under the assumed electrostatic conditions, $\vec{E}$ must be normal to the surface.

The fact that the electric field inside the conductor equals to zero has one interesting consequence that is any net charge on conductor distributes itself on the outer surface. This can be easily shown using Gauss´s law.

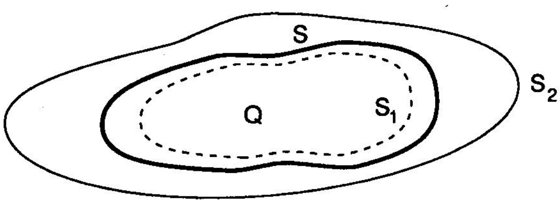

Figure 14 - 17

Consider charged conductor of any shape, such as shown in Fig. 14-17, which carries a net charge Q. Let us choose the gaussian surface S_1 shown dashed in the diagram inside the conductor.The electric field is zero at all points on this gaussian surface so the electric flux through this surface equals to zero. From the Gauss´s law, see Eq. (14-28), it is obvious that the charge inside this closed surface must be therefore equal to zero.

Now let us choose the gaussian surface S_2 outside the conductor, such as is shown in Fig. 14-17. Because this surface encloses the charge Q we can write Gauss´s law as

$$\oiint_{S_2} \vec{E}.d\vec{S} = \frac{Q}{\varepsilon_0} .$$

From the limit

$$\lim_{S_2 \to S} \oiint_{S} \vec{E}.d\vec{S} = \frac{Q}{\varepsilon_0} \quad \text{and} \quad \lim_{S_1 \to S} \oiint_{S} \vec{E}.d\vec{S} = 0$$

it is obvious that the electric charge cannot be inside but on the outer surface of the conductor only.

Gauss´s law allows us to determine the magnitude of electric field E just outside the surface of any conductor of arbitrary shape, see Fig. 14-18.

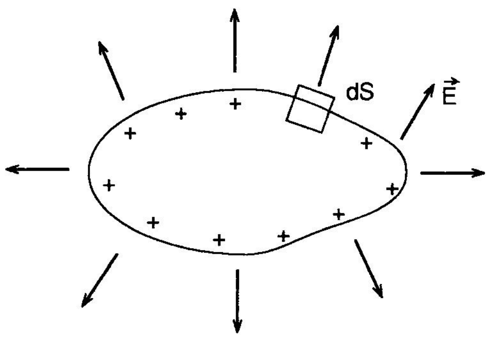

Figure 14 - 18

For this purpose we choose as our gaussian surface a small cylindrical surface, as we did in section 14-6, example 2. We choose the cylinder to be very small in height, so that one of its circular ends is just above the conductor and the other is just below the conductor's surface. The sides of the cylinder are perpendicular to the surface of conductor. The electric field is zero inside a conductor and is perpendicular to the surface just outside it. The electric flux passes only through the outside end of our cylinder. We choose the area S of the flat cylinder end small enough so that $\vec{E}$ is uniform over it. Then Gauss's law gives

$$\vec{E}.\vec{dS} = E\,dS = \frac{Q}{\varepsilon_0} = \frac{\sigma\,dS}{\varepsilon_0},$$

where σ is the surface charge density at the place of cylinder.

Thus for magnitude of electric field at surface of conductor we obtain

$$E = \frac{\sigma}{\varepsilon_0}. \qquad (14\text{-}33)$$

This is a very useful result which applies for any shape conductor.

Finally we note that, as a general rule, the surface charge density tends to be high on isolated conducting surfaces whose radii of curvature are small, and conversely. For example, the charge density tends to be relatively high on sharp points and relatively low on a plane regions on a conducting surface. The electric field $\vec{E}$ immediately above a charged surface is proportional to the surface charge density σ (see Eq. 14-33) so that $\vec{E}$ may also reach very high values near sharp points, see Figure 14-19. Corona discharges from sharp points during thunderstorms are a familiar example.

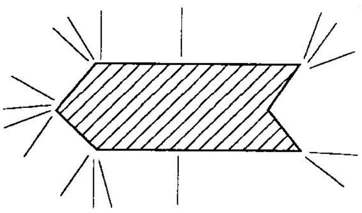

Figure 14 - 19

14 - 8 Work and Potential in an Electric Field

The electric field around charged bodies can be described not only by a vector $\vec{E}$ but also by a scalar quantity φ which is called the electric potential. To introduce the electric potential let us imagine the unit positive charge Q_0 situated in the electric field $\vec{E}$. The force $\vec{F}_e$ on this charge is given by

$$\vec{F}_{el} = Q_0\,\vec{E}.$$

The work done by this force to move the unit charge Q_0 from point K to L is:

$$W = \int_K^L \vec{F}_{el}.\vec{dr} = Q_0 \int_K^L \vec{E}.\vec{dr}.$$

Let us imagine that the electric field $\vec{E}$ is produced by a single positive point charge Q which is situated in the origin of the reference frame, see Fig. 14-20.

Electric field of a single point charge Q (see Eq. 14-5) can be written in vector notation as

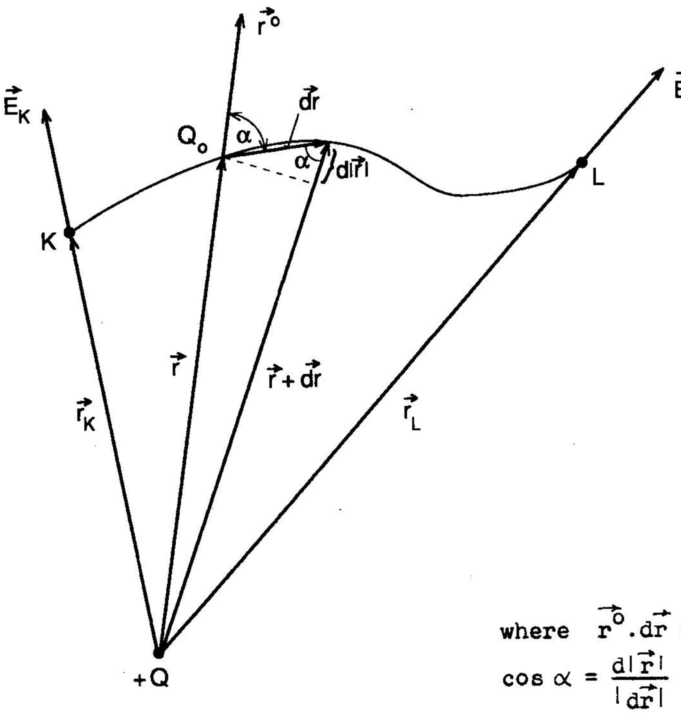

Figure 14 - 20

$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \frac{Q}{r^2} \cdot \vec{r}^o ,$$

where $\vec{r}^o$ is unit vector pointing outward from a positive charge.

In this case we can rewrite the expression for work as

$$W = Q_o \int_K^L \frac{Q}{4\pi\varepsilon_o r^2} \vec{r}_o . \vec{dr} =$$

$$= \frac{Q_o Q}{4\pi\varepsilon_o} \int_K^L \frac{dr}{r^2} = - \frac{Q_o Q}{4\pi\varepsilon_o} \left[\frac{1}{r}\right]_{r_K}^{r_L} =$$

$$= - \frac{Q_o Q}{4\pi\varepsilon_o} \left[\frac{1}{r_L} - \frac{1}{r_K}\right], \qquad (14\text{-}34)$$

where $\vec{r}^o . \vec{dr} = |\vec{r}^o| . |\vec{dr}| \cos\alpha = |\vec{dr}| \cos\alpha$ and $\cos\alpha = \frac{d|\vec{r}|}{|\vec{dr}|}$. Notice that $|\vec{dr}| \neq d|\vec{r}|$.

From Eq. (14-34) it is obvious that the work done by the electric field in moving a charge Q_o from one position to another depends only on the two positions and not on the path taken. It is also obvious when a charge Q_o is moving from one position to the same position in a closed path the work done by the electric force equals to zero.

This is a basic property of electrostatic field which we can write as

$$\oint \vec{E} . \vec{dr} = 0 . \qquad (14\text{-}35)$$

This equation states that the electric force is a conservative force, so it is possible for electrostatic field to define potential energy and potential.

The potential energy U of the unit point charge Q_o in electrostatic field is defined as work done by the external force $\vec{F}_{ext}$ in moving the charge Q_o from the reference position B to a given point P. For the sake of simplicity let us imagine that the potential energy in the reference position B equals to zero. The external force acts against the electric force so that we can write

$$\vec{F}_{ext} = - \vec{F}_{el} .$$

For potential energy we obtain

$$U = \int_B^P \vec{F}_{ext} . \vec{dr} = - \int_B^P \vec{F}_{el} . \vec{dr} = - Q_o \int_B^P \vec{E} . \vec{dr} . \qquad (14\text{-}36)$$

Electric potential (or simply potential) is defined as a potential energy per unit positive charge that is

$$\varphi = \frac{U}{Q_o} = - \int_B^P \vec{E} . \vec{dr} . \qquad (14\text{-}37)$$

We can use Eq. (14-37) to obtain the expression for potential difference between points K and L for the situation which is shown in Fig. 14-20, that is

$$- \int_K^L \vec{E} . \vec{dr} = \varphi_L - \varphi_K . \qquad (14\text{-}38)$$

For potential of the point L we obtain

$$\varphi_L = \varphi_K - \int_K^L \vec{E}.d\vec{r} . \qquad (14\text{-}39)$$

Since only the difference in the function φ at two points is ever involved we have to choose the reference point K. For convenience we will often take the reference point at infinity and we will equal potential at this point to zero. In this case for potential of the point L we have

$$\varphi_L = -\int_\infty^L \vec{E}.d\vec{r} . \qquad (14\text{-}40)$$

The electric potential can be represented graphically by drawing equipotential lines or, in three dimensions equipotential surfaces. An equipotential surface is one on which all points are at the same potential. That is, the potential difference between any two points on the surface is zero, and no work is required to move a charge from one point to another. An equipotential surface must be perpendicular to the electric field at any point. If this were not so - that is, if there were a component of $\vec{E}$ parallel to the surface - it would require work to move the charge along the surface against this component of $\vec{E}$. This would contradict the idea that it is an equipotential surface.

The potential at any point due to a group of point charges is found by

a) calculating the potential φ_i due to each charge, as if the other charges were not present and

b) adding the quantities so obtained or

$$\varphi = \sum_{i=1}^{n} \varphi_i = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \frac{Q_i}{r_i} . \qquad (14\text{-}41)$$

The sum used to calculate φ is an algebraic sum and not a vector sum like the one used to calculate $\vec{E}$ for a group of point charges (see Eq. 14-6). This is a major advantage in using electric potential.

If the charge distribution is continuous the sum in Eq. (14-41) is replaced by an integral, or

$$\varphi = \int d\varphi = \frac{1}{4\pi\varepsilon_0} \int \frac{dQ}{r} , \qquad (14\text{-}42)$$

where dQ is the element of charge, r is its distance from the point at which φ is to be calculated and $d\varphi$ is potential it establishes at that point.

Let us now find the relation between electric field and potential. For this purpose we shall consider the value of potential at two nearby points (x, y, z) and (x + dx, y + dy, z + dz). The change in φ going from the first point to the second is

$$d\varphi = \frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy + \frac{\partial\varphi}{\partial z} dz . \qquad (14\text{-}43)$$

On the other hand from the definition of potential (see Eq. 14-38) we have

$$d\varphi = -\vec{E}.d\vec{r} . \qquad (14\text{-}44)$$

The infinitesimal vector displacement $d\vec{r}$ is

$$d\vec{r} = \vec{i}\, dx + \vec{j}\, dy + \vec{k}\, dz .$$

As far as the equations (14-43) and (14-44) become identical we have

$$\frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz = - E_x\, dx - E_y\, dy - E_z\, dz$$

or

$$E_x = - \frac{\partial \varphi}{\partial x}, \quad E_y = - \frac{\partial \varphi}{\partial y}, \quad E_z = - \frac{\partial \varphi}{\partial z} .$$

Thus we can identify $\vec{E}$ as

$$\vec{E} = - \text{grad}\, \varphi . \tag{14-45}$$

The minus sign came in because the electric field points from a region of positive potential toward a region of negative potential, whereas the vector grad φ is defined so that it points in the direction of increasing φ.

The unit of electric potential is joules/coulomb (see Eq. 14-37) and is given a special name the volt (abbr. V). Note also that a positively charged object moves from the place with a high potential to a place with a low potential; a negative charge does the reverse. Potential difference, since it is measured in volts, is often referred to as voltage.

14 - 9 Electric Potential and Electric Field - Applications

In the previous section we have seen that the electric field $\vec{E}$ and electric potential φ are quantities which are intimately related and often it is a matter of convenience which one of them is used for solution of given problem.

In this section we shall show how to determine the electric potential from the known electric field and then the reverse problem, how to determine the electric field from the known potential.

Example 1: Potential Due to Single Point Charge.

Determine the potential φ at a distance r_L from a positive single point charge Q.

Solution: The electric field produced by a positive single point charge has a magnitude (see Eq. 14-4)

$$E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \tag{14-46}$$

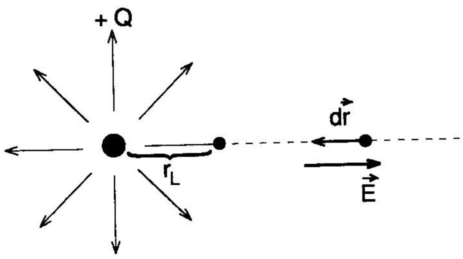

Figure 14 - 21

and is directed outward from the charge. The potential can be determined directly from Eq. (14-40), integrating along a field line from infinity to r_L, see dashed line in Fig. 14-21. We can see that $\vec{E}$ points to the right and the elementary integrating path dr, which is always in the direction of motion, points to the left. Therefore in Eq. (14-40) we have

$$\vec{E}.\vec{dr} = E \cos 180^\circ\, dr = - E.dr .$$

However as we move a distance dr to the left, we are moving in the direction of decreasing r because r is measured from Q as an origin. Thus we have to change the sign in (- dr) once more so that we obtain

$$\vec{E}.\vec{dr} = E\, dr . \tag{14-47}$$

Substituting Eq. (14-46) and Eq. (14-47) into Eq. 14-40 we have

$$\varphi_L = -\int_{\infty}^{r_L} \vec{E}.d\vec{r} = -\int_{\infty}^{r_L} E\, dr = -\int_{\infty}^{r_L} \frac{1}{4\pi\varepsilon_o} \frac{Q}{r^2} dr = \frac{Q}{4\pi\varepsilon_o r_L} \qquad (14\text{-}48)$$

This is the expression for potential at r_L relative to infinity. Note that the potential decreases with the first power of the distance whereas the electric field decreases as the square of the distance. To obtain the equipotential surfaces for a single point charge it is possible to use Eq. (14-48). We can see that the surfaces where φ = const are surfaces with constant r, therefore the equipotential surfaces are spheres concentric with the point charge.

E x a m p l e 2 : Electric Field Due to Dipole.

Calculate the electric field at any point P in xy plane due to dipole. Assume that the point P is not too close to the dipole, see Fig. 14-22.

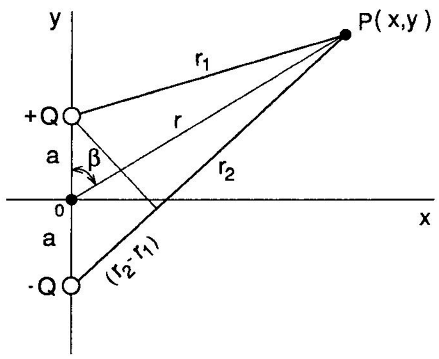

Figure 14 - 22

Solution: The electric potential at point P is the sum of potentials due to each of the two point charges (see Eq. 14-41) so

$$\varphi = \varphi_1 + \varphi_2 = \frac{1}{4\pi\varepsilon_o}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] =$$

$$= \frac{Q}{4\pi\varepsilon_o} \frac{r_2 - r_1}{r_1 r_2} .$$

We now limit our considerations to points which are distant from dipole that is $r \gg 2a$. Then we have

$$r_2 - r_1 \doteq 2a \cos\beta \quad \text{and} \quad r_1 r_2 \doteq r^2$$

so we have

$$\varphi = \frac{Q}{4\pi\varepsilon_o} \frac{2a\cos\beta}{r^2} = \frac{p}{4\pi\varepsilon_o} \frac{\cos\beta}{r^2} , \qquad (14\text{-}49)$$

where p = 2aQ is the dipole moment (see Eq. 14-19).

Now we shall calculate $\vec{E}$ as a function of position. From symmetry, $\vec{E}$, for points in the plane xy lies in this plane. We shall express it in terms of its components E_x and E_y, making use of

$$r^2 = x^2 + y^2 \quad \text{and} \quad \cos\beta = \frac{y}{\sqrt{x^2 + y^2}}$$

Thus we obtain

$$\varphi = \frac{p}{4\pi\varepsilon_o} \frac{y}{(x^2 + y^2)^{3/2}} .$$

We find E_x as a x-component of Eq. (14-15) so

$$E_x = -\frac{\partial\varphi}{\partial x} = \frac{3p}{4\pi\varepsilon_o} \frac{xy}{(x^2 + y^2)^{5/2}}$$

and similary

$$E_y = -\frac{\partial\varphi}{\partial y} = -\frac{p}{4\pi\varepsilon_o} \frac{(x^2 - 2y^2)}{(x^2 + y^2)^{5/2}} .$$

Putting y = 0 in the expression for E_y describes distant points in the median plane of the dipole and yields

$$E_y = -\frac{p}{4\pi\varepsilon_0}\frac{1}{x^3}, \qquad (14\text{-}50)$$

which agrees with the result found in Section 14-4, see Eq. (14-20). The minus sign in Eq. (14-50) indicates that $\vec{E}$ points in the negative y direction (see Fig. 14-22).

In this example we have shown that for many charge distributions, it is much easier to calculate φ first, and then $\vec{E}$ from Eq. (14-45), than to calculate $\vec{E}$ due to each charge from Coulomb's law. This is because φ due to many charges is a scalar sum, while $\vec{E}$ is a vector sum.

14 - 10 Capacitance

Let us consider two large conducting plates which are parallel to each other and separated by a distance d small compared with the plate dimensions. Let us suppose that equal and opposite charges ± Q have been put on the plates, see Fig.14-23. The charges will spread out uniformly on the inner surfaces of the plates because they will be attracted by the charges on the other plate. The plates will have surface charge density σ_+ and σ_- respectively.

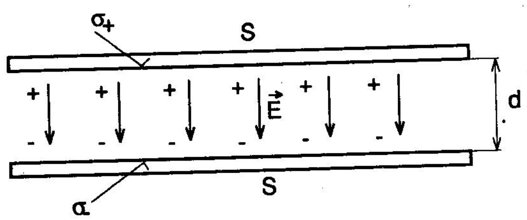

Figure 14 - 23

In this Figure S describes the area of the plates, and σS is the total charge on each plate.

From Example 2, Section 14-6, we can see that the electric field outside the plates is zero and electric field between the plates has the magnitude (see Eq. 14-32):

$$E = \frac{\sigma}{\varepsilon_0}.$$

The plates will have different potentials φ_1 and φ_2. Potential difference $(\varphi_1 - \varphi_2) = V$ is called voltage. This potential difference can be expressed as the work per unit charge required to carry a small charge from one plate to the other, so that

$$V = 1.E.d = 1\frac{\sigma}{\varepsilon_0}d = \frac{d}{\varepsilon_0}\frac{\sigma S}{S} = \frac{d}{\varepsilon_0 S}Q. \qquad (14\text{-}51)$$

We find that the voltage is proportional to the charge. Such a proportionality is found for any two conductors in space if there is a positive charge on one and a equal negative charge on the other. We can therefore write this equation of proportionality as

$$Q = CV, \qquad (14\text{-}52)$$

where C is the constant which is called capacitance and the system of two conductors is called capacitor. Thus from Eq. (14-51) for our parallel-plate capacitor we have

$$C = \frac{\varepsilon_0 S}{d}. \qquad (14\text{-}53)$$

The capacitance C is a constant for a given capacitor. Its value depends on the size, shape and relative position of the two conductors and also on the material that separates them.

A single isolated conductor can also be said to have a capacitance. In this case, C is defined as the ratio of the charge to absolute potential φ on the conductor (that is relative to $\varphi = 0$ at $r \to \infty$) so that the Eq. (14-52) remains valid.

For example, the potential of conducting sphere of radius R carrying a charge Q is

$$\varphi = \frac{1}{4\pi\varepsilon_0}\,\frac{Q}{R}\,, \qquad (14\text{-}54)$$

so its capacitance is $C = \frac{Q}{\varphi} = 4\pi\varepsilon_0 R\,.$

Note however that a single conductor is not considered a capacitor.

The SI unit of capacitance that follows from Eq. (14-52) is the coulomb/volt. A special unit, the farad (abbr. F) is used to represent it. Thus

$$1\ \mathrm{F} = \frac{1\ \mathrm{C}}{1\ \mathrm{V}}\,.$$

The submultiples of the farad, the microfarad ($1\ \mu\mathrm{F} = 10^{-6}$ F) and the picofarad ($1\ \mathrm{pF} = 10^{-12}$ F) are more convenient units in practice.

14 - 11 Electric Energy Storage

All charge configurations have a certain potential energy, equal to the work that must be done to assemble them from their initial positions originally assumed to be infinitely far apart and at rest. Let us now determine this potential energy.

Consider first the work which must be done on the system to bring some charges into a particular arrangement. Let us imagine we have two charges Q_1 and Q_2 very far apart from one another as indicated in Fig. 14-24a.

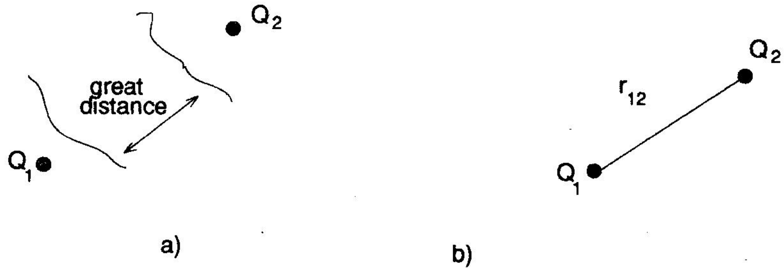

Figure 14 - 24

We shall calculate the work for moving the charges together until the distance between them is r_{12} (see Fig. 14-24b). The force that has to be applied (suppose for example that Q_1 is situated in origin of reference frame) is equal and opposite to the Coulomb force (see Eq. 14-2)

$$U_{12} = W = \int_{\infty}^{r_{1,2}} \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2}\,(-\,dr) = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{1,2}}\,. \qquad (14\text{-}56)$$

Because r is changing from infinity to $r_{1,2}$ the increment of displacement is $(-\,dr)$.

We also know from the principle of superposition that, if we have many charges present, the total force on any charge is the sum of the forces from the others. It follows, therefore that the total energy of the system of a number of charges is the sum of terms due to the mutual interaction of each pair of charges. If Q_i and Q_j are two of the charges and r_{ij} is the distance between them the energy of that particular pair is

$$U_{ij} = \frac{Q_i Q_j}{4\pi\varepsilon_o r_{ij}} .$$

The total electrostatic potential energy U_E is the sum of the energies of all possible pairs of charges, or

$$U_E = \frac{1}{2} \sum_{\substack{i,j \\ i \neq j}} \frac{Q_i Q_j}{4\pi\varepsilon_o r_{ij}} . \qquad (14\text{-}57)$$

If we have a distribution of charge specified by a charge density ρ then the charge $dQ_i = \rho_i \, dV_i$ and the charge $dQ_j = \rho_j \, dV_j$. The sum in Eq. (14-57) must be of course replaced by an integral, so we have

$$U_E = \frac{1}{2} \int_{\text{all space}} \frac{\rho_i \, \rho_j}{4\pi\varepsilon_o r_{ij}} \, dV_i \, dV_j . \qquad (14\text{-}58)$$

Notice the factor $\frac{1}{2}$, which is introduced in Eq. (14-57) and Eq. (14-58) because we have counted all pairs of charge twice. Next we notice that the integral over dV_j in Eq. (14-58) is just potential at (i), that is

$$\varphi_i = \int \frac{\rho_j \, dV_j}{4\pi\varepsilon_o r_{ij}} ,$$

so that we can write Eq. (14-58) as

$$U_E = \frac{1}{2} \int \rho_i \, \varphi_i \, dV_i$$

or since the point (j) no longer appears we can simply write

$$U_E = \frac{1}{2} \int \rho \varphi \, dV \qquad [J] \qquad (14\text{-}59)$$

This equation expresses potential energy in the case of the continuous charge distribution.

This potential energy reminds us of the potential energy stored in the gravitational field or in a compressed spring.

Similary a charged capacitor stores electrical energy. The energy stored in a capacitor is equal to work done to charge it. Charging a capacitor is to remove charge from one plate and add it to the other plate. Initially, when the capacitor is uncharged, it requires no work to move the first bit of charge over. When some charge is on each plate, it requires work to add more charge of the same sign because of the electric repulsion. The more charge already on a plate, the more work is required to add additional charge. The work needed to add a small amount of charge dq, when a potential difference V is across the plates is

$$dW = V \, dq .$$

Since $V = q/C$ at any moment, the work done is

$$W = \int_0^Q V \, dq = \frac{1}{C} \int_0^Q q \, dq = \frac{1}{2} \cdot \frac{Q^2}{C} . \qquad (14\text{-}60)$$

Taking into account that W = U and with respect to Eq. (14-52) we obtain for the energy stored in a capacitor

$$U_E = \frac{1}{2} CV^2 = \frac{1}{2} QV . \qquad (14\text{-}61)$$

Energy is not a substance and does not have a definite location. However it is reasonable to suppose that the energy stored in a capacitor resides in the electric field between the plates.

As an example let us calculate the energy stored in a parallel - plate capacitor in terms of an electric field. The electric field between the plates is approximately uniform (see Fig. 14-23) and its magnitude is (see Eq. 14-51)

$$E = \frac{V}{d} ,$$

where d is the plates separation. The capacitance for the parallel-plate capacitor is given by Eq. (14-53) as

$$C = \frac{\varepsilon_0 S}{d} .$$

Thus for potential energy we obtain

$$U_E = \frac{1}{2} CV^2 = \frac{1}{2} \frac{\varepsilon_0 S}{d} E^2 d^2 = \frac{1}{2} \varepsilon_0 E^2 (S.d) . \qquad (14\text{-}62)$$

The quantity (S.d) is the volume between the plates. If we divide both sides of Eq. (14-62) by this volume we obtain an expression for the energy per unit volume or energy density w

$$w = \frac{1}{2} \varepsilon_0 E^2 \quad [J.m^{-3}] \qquad (14\text{-}63)$$

Although we derived this equation for the special case of a parallel-plate capacitor it is true for any region of space where there is an electric field. Therefore we can say that if an electric field E exists at any point in space the energy density is proportional to the square of the electric field in this point

14 - 12 Dielectrics

In this section we shall discuss the properties of matter under the influence of the electric field. In Section 14-7 we considered the behaviour of conductors in which the charges move in response to an electric field to such points that there is no field left inside a conductor. Now we will discuss insulators, which are also called dielectrics, that is materials which do not conduct electricity.

First of all we shall try to understand in atomic terms, what happens when we place a dielectric in an electric field. There are two possibilities.

A. The molecules of some dielectric have a nonsymmetric arrangement of their atoms. For instance the water molecule H_2O has a nonsymmetric arrangement of hydrogen and oxygen atoms. There is an average positive charge on the hydrogen atoms and negative charge on the oxygen. Since the effective center of the negative charge and the effective center of the positive charge do not coincide the total charge distribution of the molecule has a dipole moment $\vec{p}$, see Fig. 14-25a. Such a molecule is called a polar molecule.

When materials, called polar, are placed in an external electric field, the electric dipole moments $\vec{p}$ tend to align themselves with an external electric field, as in Fig. 14-25b. Because the molecules are in constant thermal agitation, the degree of alignement will not be complete but will increase as the applied electric field is increased or as the temperature is decreased.

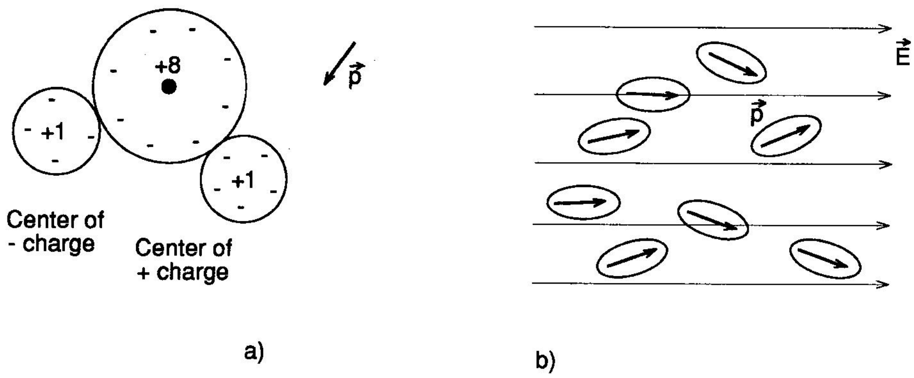

Figure 14 - 25

B. The molecules of some gases, like oxygen, which has a symmetric pair of atoms in each molecule, have no inherent dipole moment because the effective centers of the positive and negative charge are the same, see Fig. 14-26. Such molecules are called nonpolar molecules.

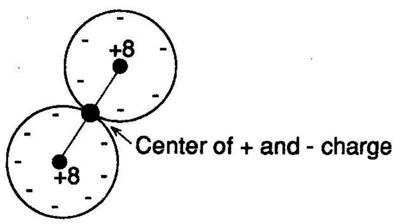

Figure 14 - 26

Let us have a look what happens if such a gas is situated in an external electric field. We shall discuss the simplest case - monoatomic gas (for instant helium). When such an atom is in an electric field, the electrons are pulled one way by the field while the nucleus is pulled the other way. Although the atoms are very stiff with respect to the electrical forces, there is a slight net displacement of the effective centers of charge, and a dipole moment is induced.

We have seen that whether or not the molecules have permanent electric dipole moments, they acquire them by induction when placed in an electric field. It is said that the dielectric becomes polarized.

We shall now study from the macroscopic view what happens if a dielectric is placed in the electric field. Faraday in 1837 first investigated the effect of filling the space between the plates of a parallel-plates capacitor with a dielectric. His experiments showed that the capacitance of such a capacitor is increased when an dielectric material is put between the plates. If the dielectric completely fills the space between the plates, the capacitance is increased by a factor ε_r which depends only on the nature of dielectric. The factor ε_r is a property of the dielectric and is called the dielectric constant. The dielectric constant of a vacuum is unity, the dielectric constant of all other dielectrics is greater than unity.

Let us imagine that we have two identical capacitors in one of which we placed a dielectric. Let the capacitance of capacitor with dielectric be C_d and the capacitance of the second capacitor be C_o. Let us place the same charge on both of them so that, with respect to Eq. (14-52), we can write

$$Q = C_d V_d = C_o V_o .$$

Let us denote

$$\varepsilon_r = \frac{C_d}{C_o} .$$

Thus

$$\frac{C_d}{C_o} = \varepsilon_r = \frac{V_o}{V_d} \quad \text{or} \quad V_d = \frac{V_o}{\varepsilon_r} \qquad (14\text{-}64)$$

We can see that the potential difference V_d between the plates of capacitor with dielectric is smaller than that for a capacitor without a dielectric by a factor $1/\varepsilon_r$. Let us now explain the above mentioned facts.

For this purpose consider a dielectric slab, showing the random distribution of positive and negative charges, see Fig. 14-27a. An external field E_o separates the center of positive charge in the slab slightly from the center of negative charge, resulting in the appearance of surface charges, see Fig. 14-27b. No net charge exists in any volume element located in the interior of the slab, see Fig. 14-27c. The surface charges set up a field $\vec{E}'$ which opposes the external field $\vec{E}_o$. Thus we can see that when a dielectric material is placed in an electric field there is a negative charge induced on one surface and positive charge induced on the other.

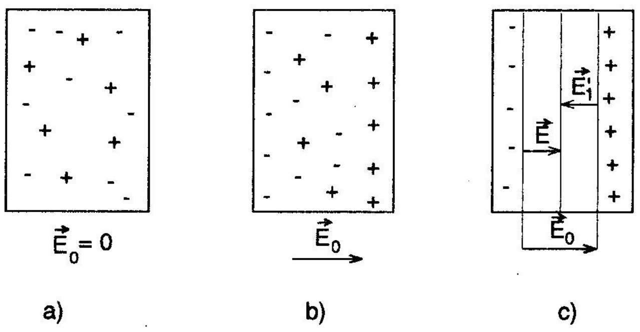

Figure 14 - 27

The positive induced surface charge must be equal in magnitude to the negative induced surface charge. Note that in this proces electrons in the dielectric are displaced from their equilibrium positions by distances that are considerably less than an atomic diameter. There is no transfer of charge over macroscopic distances.

Figure 14-27 shows that the induced surface charges will always appear in such a way that the electric field set up by them ($\vec{E}'$) opposes the external electric field $\vec{E}_o$. The resultant field $\vec{E}$ in the dielectric is the vector sum of $\vec{E}_o$ and E' that is

$$\vec{E} = \vec{E}_o + \vec{E}' . \qquad (14\text{-}65)$$

It points in the same direction as $\vec{E}_o$ but is smaller. If we place a dielectric in an electric field, induced surface charges appear which tend to weaken the original field within the dielectric.

Let us apply Gauss's law to the parallel-plate capacitor without dielectric (see Fig. 14-28a). Let the gaussian surface be shown by broken line. Thus we obtain

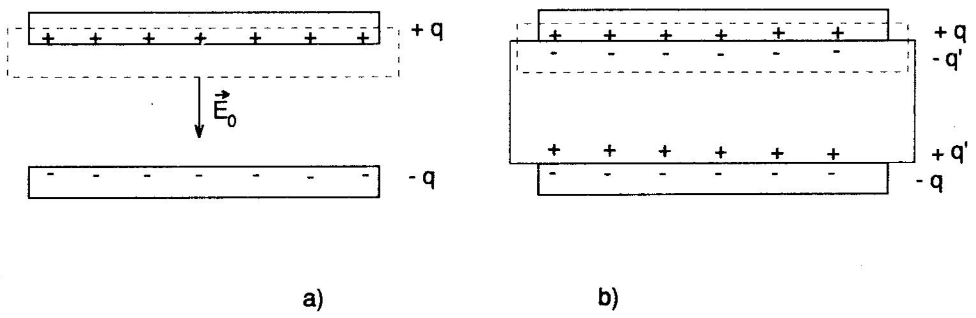

Figure 14 - 28

$$\varepsilon_0 \oiint \vec{E}_0 . d\vec{S} = \varepsilon_0 E_0 S = q$$

or

$$E_0 = \frac{q}{\varepsilon_0 S} . \tag{14-66}$$

If the dielectric is present (see Fig. 14-28b), Gauss's law gives

$$\varepsilon_0 \oiint \vec{E} . d\vec{S} = \varepsilon_0 ES = q - q' \tag{14-67}$$

or

$$E = \frac{q}{\varepsilon_0 S} - \frac{q'}{\varepsilon_0 S} \tag{14-68}$$

in which q', the induced charge, must be distinguished from q , the so - called free charge on the plates. These two charges, both of which lie within the gaussian surface, are opposite in sign, $(q - q')$ is the net charge within the gaussian surface.

As far as the relation $V = E.d$ for a parallel-plate capacitor holds whether or not dielectric is present we can write with respect to Eq. (14-64).

$$\frac{V_0}{V_d} = \frac{E_0}{E} = \varepsilon_r$$

or

$$E = \frac{E_0}{\varepsilon_r} . \tag{14-69}$$

Combining this with Eq. (14-66), we have

$$E = \frac{q}{\varepsilon_0 \varepsilon_r S} . \tag{14-70}$$

Inserting this in Eq. (14-68) yields

$$\frac{q}{\varepsilon_0 \varepsilon_r S} = \frac{q}{\varepsilon_0 S} - \frac{q'}{\varepsilon_0 S} \tag{14-71}$$

or

$$q' = q\left(1 - \frac{1}{\varepsilon_r}\right) . \tag{14-72}$$

This expression shows that the induced surface charge q' is always less in magnitude than the free charge q and is equal to zero if no dielectric is present (that is if $\varepsilon_r = 1$).

Now let us substitute in Eq. (14-67) equation (14-72). We obtain

$$\varepsilon_0 \oiint \vec{E} . d\vec{S} = q - q\left(1 - \frac{1}{\varepsilon_r}\right) .$$

After some rearrangement we obtain

$$\varepsilon_0 \oiint \varepsilon_r \vec{E} . d\vec{S} = q . \tag{14-73}$$

This important relation, although derived for a parallel-plate capacitor, is true generally.

Note the following:

1) The flux integral now contains a dielectric constant ε_r .

2) The charge q contained within the gaussian surface is taken to be the free charge only. Induced surface charge is deliberately ignored on the right side of this equation, having been taken into account by the introduction of ε_r on the left side. Equations (14-67) and (14-73) are completely equivalent formulations.

Let us rewrite now Eq, (14-71), which applies to a parallel-plate capacitor containing a dielectric, as

$$\frac{q}{S} = \varepsilon_o\left(\frac{q}{\varepsilon_o \varepsilon_r S}\right) + \frac{q'}{S} . \qquad (14\text{-}74)$$

The quantity in parentheses (see Eq. 14-70) is the electric field $\vec{E}$ in the dielectric. The last term in Eq. (14-74) is the induced surface charge per unit area. We call it the electric polarization P, or

$$P = \frac{q'}{S} . \qquad (14\text{-}75)$$

The electric polarization P can be defined in an equivalent way by multiplying the numerator and denominator in Eq. (14-75) by d , the thickness of dielectric, or

$$P = \frac{q'd}{S\,d} . \qquad (14\text{-}76)$$

The numerator is the product (q'd) of the magnitude of the polarization charges by their separation. It is thus the induced electric dipole moment of the dielectric. Since the denominator (S d) is the volume of the dielectric, we see that the electric polarization can also be defined as the induced electric dipole moment per unit volume of the dielectric. This definition suggests that since that the electric dipole moment is a vector the electric polarization is also a vector, its magnitude being P. The direction of $\vec{P}$ is from the negative induced charge to the positive induced charge, as for any dipole.

We can now rewrite Eq. (14-74) as

$$\frac{q}{S} = \varepsilon_o E + P . \qquad (14\text{-}77)$$

The quantity on the right occurs so often in electrostatic problems that we give it a special name electric displacement D, or

$$D = \varepsilon_o E + P \qquad (14\text{-}78)$$

in which

$$D = \frac{q}{S} . \qquad (14\text{-}79)$$

Note that the units for P and D are $[C.m^{-2}]$.

Since $\vec{E}$ and $\vec{P}$ are vectors, $\vec{D}$ must also be one, so that we have

$$\vec{D} = \varepsilon_o\vec{E} + \vec{P} . \qquad (14\text{-}80)$$

In Fig. 14-29 are shown vectors $\vec{D}$, $\varepsilon_o\vec{E}$ and $\vec{P}$ in the dielectric (upper right) and in the gap (upper left) for a parallel plate capacitor with a dielectric. In

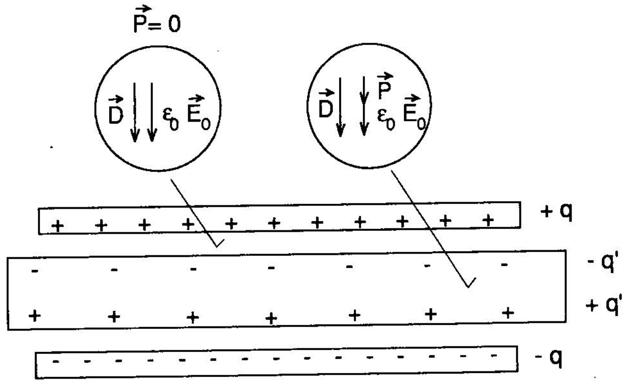

Figure 14 - 29

Fig. 14-30 are shown samples of the lines which are associated with $\vec{D}$, $\varepsilon_0\vec{E}$ and $\vec{P}$.

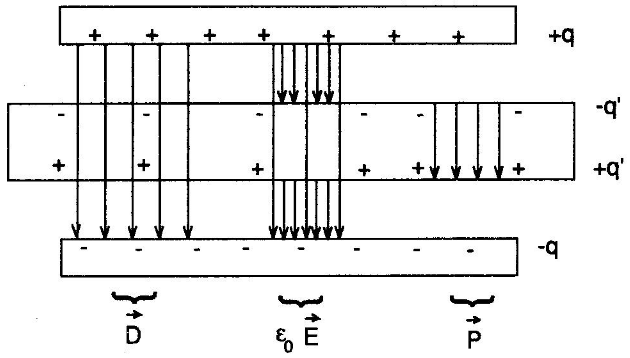

Figure 14 - 30

From the definitions we see the following:

1. Vector $\vec{D}$ (see Eq. 14-79) is connected with the free charge only. We can represent the vector field of $\vec{D}$ by lines of $\vec{D}$, just as we represent the field of $\vec{E}$ by lines of force. Figure 14-29 shows that the lines $\vec{D}$ begin and end on free charges.
2. Vector $\vec{P}$ is connected with the polarization charge only. It is also possible to represent this vector field by lines. Figure 14-29 shows that thé lines of $\vec{P}$ begin and end on the polarization charges.
3. Vector $\vec{E}$ is connected with all charges that are actually present, whether free or polarization. The lines of $\vec{E}$ reflect the presence of both kinds of charge.
4. Vector $\vec{P}$ vanishes outside the dielectric, $\vec{D}$ has the same value in the dielectric and in the gap and $\vec{E}$ has different values in the dielectric and in the gap.

The vectors $\vec{D}$ and $\vec{P}$ can both be expressed in terms of $\vec{E}$ alone. So, putting Eq. (14-66) into Eq. (14-69) and with respect to Eq. (14-79) we obtain, extended to a vector form

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} . \qquad (14\text{-}81)$$

We can also write the polarization (see Eq. 14-75 and Eq. 14-72) as

$$P = \frac{q'}{S} = \frac{q}{S}\left(1 - \frac{1}{\varepsilon_r}\right) .$$

Since $\frac{q}{S}$ is D we can rewrite this, using Eq. (14-81) and casting the result into vector form as

$$\vec{P} = \varepsilon_0 (\varepsilon_r - 1) \vec{E} . \qquad (14\text{-}82)$$

This shows clearly that in vacuum ($\varepsilon_r = 1$) the polarization vector $\vec{P}$ is zero.

The definition of $\vec{D}$ given by Eq. (14-81) allows us to write Eq. (14-73), that is Gauss's law in the presence of dielectric, simply as

$$\oint\!\!\oint \vec{D}.d\vec{S} = q , \qquad (14\text{-}83)$$

where q represents the free charge only.

Finally it must be pointed out that the vector $\vec{E}$ is, still, the basic electric field vector. The vectors $\vec{P}$ and $\vec{D}$ are useful auxiliaries for more advanced work.