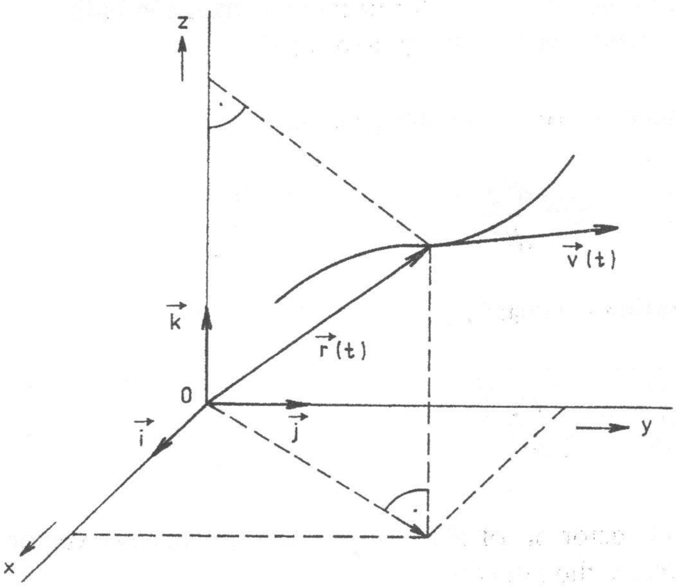
1. MECHANICS

1.1 KINEMATICS

Particle motion description



The **path** of a motion is described if, for every moment of the motion, we know the coordinates x, y, z of the moving particle. Alternatively, we can say that the path is described if we know the **position vector** of the particle as a function of time:

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$

where **i**, **j**, **k** are unit vectors whose magnitude is one and which point along the chosen co-ordinate axes as shown in Fig.

The **velocity vector** **v** of a motion is defined as the first derivative of the position vector with respect to time:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$$

It represents the rate at which the position vector of the particle changes with time. The direction of **v** at any moment is along a line tangent to the path at that moment. The **SI unit of velocity** is meter per second: m s^{-1} .

Since the components of the velocity vector are given as

$$v_x = \frac{dx}{dt} \qquad v_y = \frac{dy}{dt} \qquad v_z = \frac{dz}{dt}$$

the **magnitude of the velocity** is equal to

$$v = |\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

The **acceleration vector** $\mathbf{a}$ is defined as the first derivative of velocity vector $\mathbf{v}$ with respect to time or the second derivative of position vector $\mathbf{r}$ with respect to time:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}$$

It represents the rate at which the velocity vector changes with time. Thus, the acceleration can result from either a change in the magnitude of the velocity vector or from a change in the direction of the velocity vector or from a change in both.
The **SI unit of acceleration** is meter per second squared: m s^{-2} .

Since the components of the acceleration vector are given as

$$a_x = \frac{d^2x}{dt^2} \qquad a_y = \frac{d^2y}{dt^2} \qquad a_z = \frac{d^2z}{dt^2}$$

the **magnitude of the acceleration** is equal to

$$a = |\mathbf{a}| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$$

If we are given the acceleration vector $\mathbf{a}$ of the motion, then the velocity vector and the position vector are calculated as the integrals

$$\mathbf{v}(t) = \int_0^t \mathbf{a}\, dt + \mathbf{v}_0 \qquad ; \qquad \mathbf{r}(t) = \int_0^t \mathbf{v}\, dt + \mathbf{r}_0$$

where the constants of integration $\mathbf{r}_0$ and $\mathbf{v}_0$ represent the position vector and the velocity vector, respectively, at the beginning of the motion at time t = 0.

For uniformly accelerated motion, the magnitude of the acceleration is constant and these integrals can be calculated very easily and thus

$$\mathbf{v}(t) = \mathbf{a}\, t + \mathbf{v}_0$$

and

$$\mathbf{r}(t) = \tfrac{1}{2}\mathbf{a}t^2 + \mathbf{v}_0 t + \mathbf{r}_0$$

If this motion is in a straight line all vectors in these expressions are parallel and thus the expressions for **v** and **r** can be written in the scalar form.

The most common example of motion with (nearly) constant acceleration is that of a body falling toward the earth. In the absence of air resistance we find that all bodies, regardless of their size and weight fall with the same acceleration at the same point on the earth's surface. Such an acceleration is called **acceleration due to gravity** and is denoted by the vector **g**. Near the earth's surface its magnitude is approximately 9.8 ms^{-2} and it is directed down toward the centre of the earth.

For the velocity vector of this motion

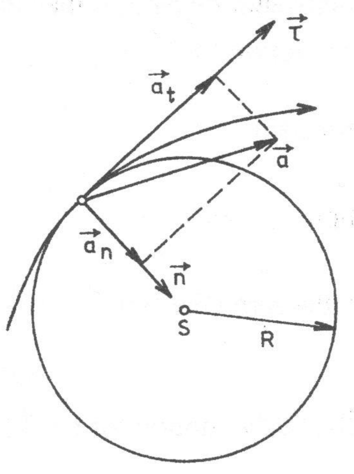
$$\mathbf{v}(t) = \int_0^t \mathbf{g}dt = \mathbf{g}t + \mathbf{v}_0$$

where the constant of integration $\mathbf{v}_0$ is an initial velocity vector at time t = 0.
The position vector of this motion is

$$\mathbf{r}(t) = \int_0^t \mathbf{v}dt = \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{v}_0 t + \mathbf{r}_0$$

where $\mathbf{r}_0$ represents the position vector for time t = 0.

Resolution of the acceleration vector



If a velocity vector is changing there will be tangential acceleration $\mathbf{a}_t$ as well as radial (centripetal) acceleration $\mathbf{a}_n$.

a) **Tangential acceleration** arises from a change in the magnitude of the velocity vector, and has the magnitude

$$a_t = \frac{dv}{dt}$$

The tangential acceleration always points in a direction tangent to the trajectory and is in the direction of motion (parallel to **v**) if the velocity is increasing. If the velocity is decreasing it points anti-parallel to **v**. In either case, $\mathbf{a}_t$ *and* $\mathbf{a}_n$ are always perpendicular to each other.

b) **Radial (centripetal) acceleration** arises from a change in the direction of the velocity vector and has the magnitude

$$a_n = \frac{v^2}{r}$$

where *r* is the radius of curvature .

c) The total vector acceleration **a** is the sum of these two: $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n$

d) Since $\mathbf{a}_n$ *and* $\mathbf{a}_t$ are always perpendicular to each other, the magnitude of total acceleration **a** at any moment is

$$a = \sqrt{a_t^2 + a_n^2}$$

e) It is necessary to maintain the difference between the magnitude of the acceleration vector

$$a = |\mathbf{a}| = \left|\frac{\mathbf{dv}}{\mathbf{dt}}\right| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}$$

and the magnitude of the tangential acceleration vector

$$a_t = \frac{dv}{dt} = \frac{d|\mathbf{v}|}{dt} = \frac{d}{dt}\left(\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right)$$

Quantities describing motion in a straight line

a) uniform motion (a=0)

the path s [m] : $s = v_0 t + s_0$ (v_0, s_0 are the velocity and the path, respectively, at the beginning of the motion for t=0)

the velocity $v\left[ms^{-1}\right]$: $v = v_0$; the velocity is constant

the acceleration $a\left[ms^{-2}\right]$: $a = 0$; the acceleration is zero

b) uniformly accelerated motion (a is equal to the non-zero constant)

$$s = \tfrac{1}{2}at^2 + v_0 t + s_0$$

(v_0 ; s_0 are the velocity and the path, respectively, of the motion for t = 0)

$$v = at + v_0$$

(the velocity increases at a uniform rate; v_0 is the velocity at the beginning of the motion)

$$a = \text{constant} \neq 0$$

(the acceleration is constant; it does not depend on time)

c) non-uniformly accelerated motion (a is not constant)

$$s = \int_0^t v\,dt$$

$$v = \frac{ds}{dt} \quad ; \quad v = \int_0^t a\,dt$$

$$a = \frac{dv}{dt} \quad ; \quad a = \frac{d^2 s}{dt^2}$$

Motion with acceleration due to gravity

a) Free fall: from expressions for straight, uniformly accelerated motion if we put

$s_0 = 0, \quad v_0 = 0, \quad a = g$

$$s = \frac{1}{2} g t^2 \quad , \quad v = gt$$

b) Projection vertically down: from expressions for straight, uniformly accelerated motion if we put $s_0 = 0, \quad a = g$

$$s = \frac{1}{2} g t^2 + v_0 t \quad , \quad v = gt + v_0$$

, where v_0 is an initial velocity of the projection

c) Projection vertically up: from expressions for straight, uniformly accelerated motion if we put $s_0 = 0, \quad a = -g$

$$s = v_0 t - \frac{1}{2} g t^2 , \quad v = v_0 - gt$$

where v_0 is the initial velocity of the projection

Quantities describing motion in a circle

Angular velocity ω is defined as the rate of the change in an angular displacement

$$\omega = \frac{d\varphi}{dt}$$

The **dimension of angular velocity** is s^{-1}.

Angular acceleration ε is defined as the rate of the change in an angular velocity

$$\varepsilon = \frac{d\omega}{dt}$$

The **dimension of angular acceleration is** s^{-2}**.**

Angular velocity ω can be related to linear velocity: $v = r\frac{d\varphi}{dt} = r\,\omega$.

Thus, magnitude of the linear velocity of motion in a circle is equal to the radius r of the circle times the magnitude of the angular velocity

Angular acceleration ε is related to tangential acceleration a_t :

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\,\varepsilon$$

Radial acceleration can be related to angular velocity:

$$a_n = \frac{v^2}{r} = r\,\omega^2$$

a) Uniform circular motion ($\varepsilon = 0$)

angular displacement φ: $\varphi = \omega_0 t + \varphi_0$

where ω_0, φ_0 are the angular velocity and the angle respectively, at the beginning of the motion

angular velocity $\omega\left[s^{-1}\right]$: ω = constant; the angular velocity is constant

angular acceleration $\varepsilon\left[s^{-2}\right]$: $\varepsilon = 0$; the angular acceleration is zero

period T [s]

frequency $f\left[s^{-1}\right]$: $f = \frac{1}{T}$

angular velocity $\omega\left[s^{-1}\right]$: $\omega = \frac{2\pi}{T} = 2\pi f$

linear velocity $v = \left[ms^{-1}\right]$: $v = \frac{2\pi r}{T} = 2\pi f r = \omega r$, where r is the radius of the circle; the linear velocity is constant

radial acceleration $a_n\left[ms^{-2}\right]$: $a_n = \frac{v^2}{r} = \omega^2 r = 4\pi^2 f^2 r = \frac{4\pi^2}{T^2} r$

tangential acceleration $a_t\left[ms^{-2}\right]$: $a_t = \frac{dv}{dt} = 0$

b) Uniformly accelerated motion (ε = constant $\neq 0$)

$\varphi = \frac{1}{2}\varepsilon t^2 + \omega_0 t + \varphi_0$ (ω_0, φ_0 are the angular velocity and the angle, respectively, at the beginning of the motion for $t = 0$)

$\omega = \varepsilon t + \omega_0$ (the angular velocity increases linearly with time; ω_0 is the angular velocity at the beginning of the motion)

ε = constant $\neq 0$ (the angular acceleration is constant; it does not depend on time)

c) Non-uniformly accelerated motion ($\varepsilon \neq$ constant)

$$\varphi = \int_{t_1}^{t_2} \omega dt$$

$$\omega = \frac{d\varphi}{dt} \quad ; \quad \omega = \int_0^t \varepsilon dt$$

$$\varepsilon = \frac{d\omega}{dt} \quad ; \quad \varepsilon = \frac{d^2\varphi}{dt^2}$$

Problem 1-1. If the position of a particle with respect to some origin is given by the position vector

$$\mathbf{r(t)} = t^2\mathbf{i} - 4t\mathbf{j} + (2 + t^3)\mathbf{k}$$

find its velocity and acceleration vectors.

Solution: The velocity components can be found by differentiating each component of $\boldsymbol{r}$ separately to obtain

$$v_x = \frac{d(t^2)}{dt} = 2t \qquad v_y = \frac{d(-4t)}{dt} = -4 \qquad v_z = \frac{d(2+t^3)}{dt} = 3t^2$$

The velocity is then the vector sum of these components:

$$\mathbf{v} = 2t\mathbf{i} - 4\mathbf{j} + 3t^2\,\mathbf{k}$$

To find the acceleration vector, we take derivatives of the components of $\boldsymbol{v}$ with respect to time:

$$a_x = \frac{d(2t)}{dt} = 2 \qquad a_y = \frac{d(-4)}{dt} = 0 \qquad a_z = \frac{d(3t^2)}{dt} = 6t$$

Their vector sum is the acceleration vector $\mathbf{a} = 2\mathbf{i} + 6t\mathbf{k}$, which has no y component; its x component is constant.

Problem 1-2. The motion of a particle is described by the vector equation

$$\mathbf{r}(t) = (2t+5)\mathbf{i} - t^2\mathbf{j} + \tfrac{1}{3}t^3\mathbf{k} \qquad [m]$$

where the parameter t is time.

Determine for any time t:

a) its co-ordinates and distance from the origin,

b) velocity and acceleration vectors and their magnitudes,

c) the tangential acceleration and the radial acceleration.

Solution:

a) $x(t) = 2t + 5 \qquad y(t) = -t^2 \qquad z(t) = \frac{1}{3}t^3$

these formulas represent so called parametric equations of motion.

The distance from the origin at any time is given by the magnitude of the position vector (its absolute value), so

$$r(t) = \sqrt{x^2 + y^2 + z^2} = \sqrt{(2t+5)^2 + t^4 + \tfrac{1}{9}t^6} \qquad [m]$$

b)

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = 2\mathbf{i} - 2t\mathbf{j} + t^2\mathbf{k} \qquad [ms^{-1}]$$

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -2\mathbf{j} + 2t\mathbf{k} \qquad [ms^{-2}]$$

$$v(t) = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{4 + 4t^2 + t^4} = t^2 + 2 \quad [ms^{-1}]$$

$$a(t) = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2} \quad [ms^{-2}]$$

c)

$$a_t = \frac{dv}{dt} = 2t \qquad [ms^{-2}]$$

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{4 + 4t^2 - 4t^2} = 2 \qquad [ms^{-2}]$$

Note that the radial acceleration of this motion does not depend on time. It is constant for all time of the motion.

For example, at time t = 3s the position of the particle will be given by co-ordinates x = 11m, y = - 9 m , z = 9 m and its distance from the origin will be $r = 16.8$ m.

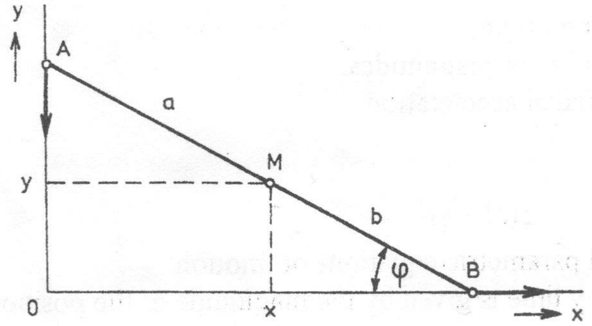
The velocity and the acceleration of the particle will have magnitudes $11ms^{-1}$ *and* $6.3ms^{-2}$, respectively, at time t = 3s .

The tangential acceleration of the particle at time t = 3s will be equal to $6ms^{-2}$ and the radial acceleration has the magnitude $2ms^{-2}$ which is constant for all time of the motion.

If we take an interest in the path travelled, for example, between the instants $t_1 = 3s$ *and* $t_2 = 6s$, we have to calculate the integral

$$s = \int_{t_1}^{t_2} v(t)\,dt = \int_3^6 (t^2 + 2)\,dt = 69\,m$$

Problem 1-3. The end point A of the bar in Fig. may move along the y axis, and end point B may move along the x axis of the rectangular co-ordinate system.



Determine the trajectory of the point M of the bar if the motion of the bar starts from its vertical position and finishes when the bar reaches the horizontal position.

<u>Solution:</u> The instantaneous position of point M is given by its co-ordinates:

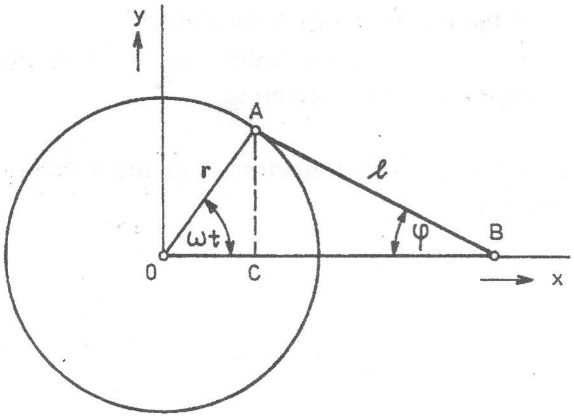
$$x = a\cos\varphi, \quad y = b\sin\varphi$$

$$or \quad \frac{x}{a} = \cos\varphi\,, \quad \frac{y}{b} = \sin\varphi$$

If we square and add both equations we have $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Thus, the point M moves from the initial position (0,b) to the final position (a,0) in an ellipse with axes $2a$ and $2b$.

Problem 1-4. The end point B of a bar whose length is l may move along the x axis only. The joint point A moves with a constant angular velocity ω in a circle of radius $r < l$. Examine the motion of the end point B.



Solution: The instantaneous position of point B at any time is given

$$x_B(t)=\overline{OC}+\overline{CB}=r\cos\omega t+l\cos\varphi.$$

From the sine theorem

$$\frac{\sin\varphi}{\sin\omega t}=\frac{r}{l}$$

$$\sin\varphi=\frac{r}{l}\sin\omega t \quad \Rightarrow \quad \cos\varphi=\sqrt{1-\frac{r^2}{l^2}\sin^2\omega t}$$

Thus, $$x_B(t)=r\cos\omega t+\sqrt{l^2-r^2\sin^2\omega t}$$

Problem 1-5. The acceleration of a linear motion increases at a uniform rate. The motion starts from rest and at time $t_1=90\,s$ the magnitude of its acceleration equals $a_1=0.5\,ms^{-2}$.

Calculate: a) the dependence of the velocity and the trajectory on time,

b) the velocity and the trajectory travelled at time t = 90 s.

Solution: a) The motion is non uniform, but its acceleration varies uniformly with time, so we may write for it

$$a(t)=k\,t,$$

where coefficient k can be determined from given values : $k=\frac{a_1}{t_1}$

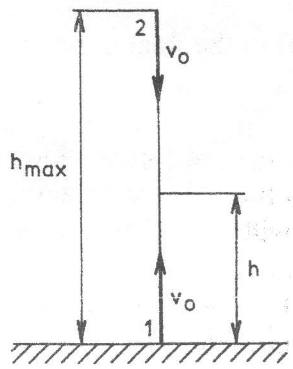
Then

$$v(t)=\int_0^t a(t)dt=\int_0^t \frac{a_1}{t_1}t\,dt=\frac{a_1}{2t_1}t^2$$

and $$s(t)=\int_0^t v(t)dt=\int_0^t \frac{a_1}{2t_1}t^2dt=\frac{a_1}{6t_1}t^3$$

b) For t = 90 s : $v=22.5\,ms^{-1}$ *and* $s=675\,m$

Problem 1-6. A body makes a projectile motion vertically up with an initial velocity $v_0 = 4.9 ms^{-1}$. At the same time from the top of its trajectory the other body starts to make a free fall with the same initial velocity v_0.



Determine: a) the time t_0 at which the bodies meet,

b) the distance h from the surface of the Earth at which they meet,

c) the velocities of the bodies at the moment of their meeting.

Solution: The trajectory and the velocity of the first body are described by the expressions

$$s_1 = v_0 t - \frac{1}{2} g t^2 , \qquad v_1 = v_0 - gt$$

and for the other body

$$s_2 = v_0 t + \frac{1}{2} g t^2 , \qquad v_2 = v_0 + gt$$

The first body will reach the top of its trajectory at time t_m which is calculated from the condition $v_1 = 0$;

hence $$t_m = \frac{v_0}{g}$$

and the top of its trajectory is equal to

$$h_{max} = \frac{v_0^2}{g} - \frac{v_0^2}{2g} = \frac{v_0^2}{2g}$$

At the moment of the meeting their trajectories follow the condition

$$s_1 + s_2 = h_{max}$$

and thus $$v_0 t_0 - \frac{1}{2} g t_0^2 + v_0 t_0 + \frac{1}{2} g t_0^2 = \frac{v_0^2}{2g}$$

From this equation the time of the meeting equals

$$t_0 = \frac{v_0}{4g} = 0.125 s.$$

The distance from the surface of the Earth at which the bodies meet is equal to

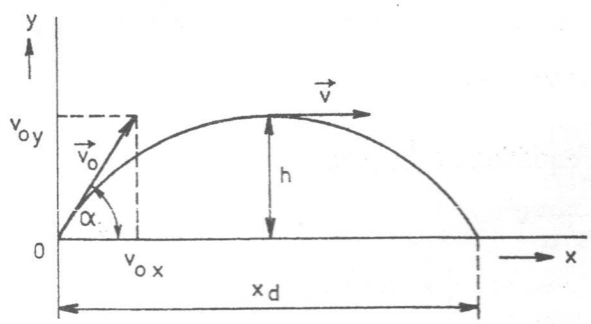
$$h = v_0 t_0 - \frac{1}{2} g t_0^2 = \frac{7 v_0^2}{32 g} = 0.53 m$$

and the velocities of the bodies at the moment of the meeting

$$v_1 = \frac{3}{4} v_0 = 3.67\, ms^{-1}$$

$$v_2 = \frac{5}{4} v_0 = 6.12\, ms^{-1}$$

Problem 1-7. Investigate the motion of a particle in the gravitation field under the following conditions.



At the beginning of the motion its position vector $\mathbf{r} = 0$ with respect to the chosen co-ordinate system and its initial velocity vector $\mathbf{v_0}$ makes an angle α with the horizontal (see Fig.)

Determine: a) the time dependence of the particle's velocity

b) the parametric equations of the motion

c) the equation of the particle's trajectory

Solution: The particle moves in a curved path with the constant acceleration of gravity vector $\mathbf{g} = -g\,\mathbf{j}$ in the chosen co-ordinate system (see Fig.).

The velocity vector of the motion is

$$\mathbf{v} = \int_0^t \mathbf{g}\,dt = \mathbf{g}t + \mathbf{v}_0 \qquad (I)$$

where $\mathbf{v}_0$ is the initial velocity vector at time t = 0.

The position vector of the particle is

$$\mathbf{r} = \int_0^t \mathbf{v}\,dt = \int_0^t (\mathbf{g}t + \mathbf{v}_0)dt = \frac{1}{2}\mathbf{g}t^2 + \mathbf{v}_0 t + \mathbf{r}_0 \qquad (II)$$

where $\mathbf{r}_0$ is the position vector of the particle at time t=0.

With respect to initial conditions we write

$$\mathbf{r}_0 = 0 \qquad or \qquad x_0 = y_0 = 0$$

$$v_{0x} = v_0 \cos\alpha, \quad v_{0y} = v_0 \sin\alpha$$

$$\mathbf{g} = (0, -g)$$

From equation (I) we have the components of the velocity vector

$$v_x = v_0 \cos\alpha$$

$$v_y = -gt + v_0 \sin\alpha$$

and the magnitude of the velocity will be given as

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{v_0^2 - 2v_0 gt\sin\alpha + g^2t^2}$$

where v_0 is the magnitude of the velocity vector at the beginning.

From equation (II) we can write the components of the position vector of the particle

$$x = v_0 t \cos\alpha$$

$$y = -\frac{1}{2} g t^2 + v_0 t \sin\alpha$$

These are the parametric equations of the motion.

Eliminating parameter t we have the equation of the path

$$y = x\, tg\alpha - \frac{g}{2v_0^2 \cos^2\alpha} x^2$$

This is the equation of a parabola.

Note that for $\alpha = 0$ the motion will be a horizontal projectile motion, for $\alpha = \frac{\pi}{2}$ it will be a projectile motion vertically upward and for $0 \langle \alpha \langle \pi/2$ it will be a projectile motion at an angle.

Problem 1-8. Refer to results of the previous problem, and calculate:

a) the time t_1 taken by the particle to reach the top of its trajectory,
b) the maximum height reached by the particle,
c) the time t_2 required to hit ground level
d) the horizontal range at which the particle hits ground level

Solution: a) from the condition $v_y = 0$. $\quad t_1 = \frac{v_0 \sin\alpha}{g}$

b) substitute time t_1 into the expression for the y co-ordinate:

$$h_{max} = \frac{v_0^2 \sin^2\alpha}{2g}$$

c) from the condition $y = 0$: $\quad t_2 = \frac{2v_0 \sin\alpha}{g} = 2t_1$

d) substitute time t_2 into the expression for the x co-ordinate:

$$x_d = \frac{v_0^2 \sin 2\alpha}{g}$$

Note that a projection angle of 45^0 gives the maximum horizontal range.

Problem 1-9. The angle φ of a particle moving in a circle depends on time as follows

$$\varphi(t) = k_1 t + k_2 t^3$$

where $k_1 = \frac{\pi}{10} s^{-1}$, $k_2 = \frac{\pi}{40} s^{-3}$

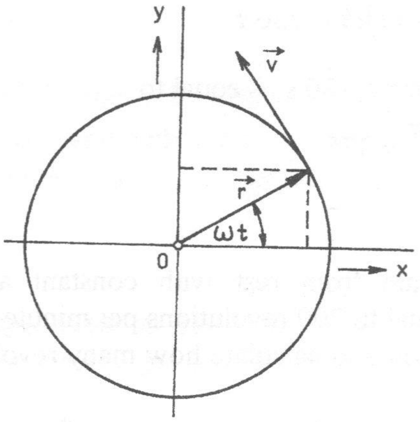
Determine the dependence of the angular velocity ω as well as the angular acceleration ε of the motion at time t

Solution: Using the definitions for ω and ε we can write:

$$\omega(t) = \frac{d\varphi}{dt} = k_1 + 3k_2t^2$$

$$\varepsilon(t) = \frac{d\omega}{dt} = 6k_2t$$

Problem 1-10. A particle moves uniformly in a circle of radius *r* with angular velocity ω.



Answer the following questions: what are its position vector, the velocity vector, the acceleration vector, and the magnitude of its radial and tangential acceleration?

Solution: The co-ordinates of the particle are given as follows

$x = r\cos\omega t$, $y = r\sin\omega t$

Hence, the position vector of the particle equals

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = r(\mathbf{i}\cos\omega t + \mathbf{j}\sin\omega t)$$

Then the velocity vector equals

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = -\mathbf{i}r\omega\sin\omega t + \mathbf{j}r\omega\cos\omega t$$

And the acceleration vector equals

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = -r\omega^2(\mathbf{i}\cos\omega t + \mathbf{j}\sin\omega t) = -\omega^2\mathbf{r}$$

It is clear that the acceleration vector has a direction opposite to that of the position vector. It points toward the centre and thus it is identical with the vector of the centripetal acceleration. In view of $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n$ the vector of the tangential acceleration must be equal to zero.

Thus, the magnitude of the total acceleration is equal to that of the centripetal acceleration.

Hence, $a = a_n = |\mathbf{a}| = \omega^2 r$

Problem 1-11. A flywheel rotates with frequency n=1500 revolutions per minute. Due to braking, the motion becomes uniformly retarded, and it finishes during time t_0=30 s after the braking started. Determine the angular acceleration and the number of revolutions performed from the beginning of braking till the motion stops.

Solution: An instantaneous value of the angular velocity of a uniformly accelerated circular motion is given by $\omega(t) = \omega_0 + \varepsilon t$, where ω_0 is the initial angular velocity of the motion.

For our case $\omega_0 = 2\pi f = 2\pi \frac{1500}{60} = 50\pi \quad (s^{-1})$

For time t_0 ω(t) must be zero, so $\omega(t_0) = \omega_0 + \varepsilon t_0 = 0$

Thus the angular acceleration is equal to $\varepsilon = -\frac{\omega_0}{t_0} = -\frac{50\pi}{30} = -\frac{5}{3}\pi \quad (s^{-2})$

The angle subtended during time t_0 is equal to

$$\varphi_0 = \int_0^{t_0} \omega(t)dt = \int_0^{t_0} (\omega_0 + \varepsilon t)dt = 750\pi$$

The number of revolutions performed during time t_0=30 s is equal to

$$N = \frac{\varphi_0}{2\pi} = \frac{750\pi}{2\pi} = 375$$

Problem 1-12. A disc wheel starts to rotate from rest with constant angular acceleration. At time t=20 s its frequency is equal to 200 revolutions per minute. Determine the angular acceleration of the motion and calculate how many revolutions the wheel will make in time t=20 s ?

Solution: The angular velocity of the wheel at time t=20 s is equal to ω=2πn , where $n = \frac{200}{60}$ represents the number of revolutions per second.

Thus, the angular velocity at time t=20 s is equal to ω=2πn=21 s^{-1}.

Since the motion starts from rest the angular acceleration equals

$$\varepsilon = \frac{\omega}{t} = \frac{21}{20} = 1.05\ s^{-2}$$

The angle subtended in time t=20 s equals $\varphi(t) = \int_0^t \omega(t)dt = \int_0^t \varepsilon t\, dt = \frac{1}{2}\varepsilon t^2 = 210$

Thus, the number of revolutions in time t=20 s is equal to $N = \frac{210}{2\pi} = 33.4$

Problem 1-13. What is the initial velocity of a uniformly retarded motion whose acceleration is $a = -1.2 ms^{-2}$ if a path of 135 m is travelled before the motion stops?

$[v_0 = 18 ms^{-1}]$

Problem 1-14. What is the velocity of a uniformly accelerated motion at time t=10 s if the path travelled in time $t_1 = 25 s$ is 110 m ? The initial velocity of the motion was zero.

$[v = 3.2 ms^{-1}]$

Problem 1-15. What is the acceleration of a uniformly retarded motion if its velocity decreases from $60\,km\,h^{-1}$ *to* $40\,km\,h^{-1}$ on a path of 110 m?

$$\left[a = -0.70\,ms^{-2}\right]$$

Problem 1-16. Two objects move along a straight line in opposite directions. Their accelerations are $a_1 = 6\,ms^{-2}, a_2 = 4\,ms^{-2}$ and their initial velocities are $v_{01} = 10\,ms^{-1}, v_{02} = 15\,ms^{-1}$. Their separation at the beginning is 750 m. In what time will the objects meet?

[t = 10 s]

Problem 1-17. Two bodies start to move along a straight line in opposite directions from a distance of 100 m. The first one makes a uniform motion with velocity $v = 3\,ms^{-1}$, the other makes a uniformly accelerated motion with acceleration $a = 4\,ms^{-2}$ and with initial velocity $v_0 = 7\,ms^{-1}$.
Determine the time and the position of their meeting.

[t = 5 s]
[15 m from the initial position of the first body]

Problem 1-18. A free falling body covered the last h meters of its path in time t. Calculate the height h_x from which it started to fall .

$$\left[h_x = \frac{h^2}{2gt^2} + \frac{h}{2} + \frac{gt^2}{8}\right]$$

Problem 1-19. A body falls toward the earth from a height of 195 m. At the same time another body starts to make a projection vertically up from the earth's surface with initial velocity $v_0 = 65\,ms^{-1}$.
Determine the time and the height of their meeting. Consider g = 9.81 ms^{-2} .

[t = 3 s ; h = 151 m]

Problem 1-20. Calculate the initial velocity and the maximum height of a body making a projection vertically up if it returned to the earth's surface in time 20 s.

$$\left[v_0 = 98\,ms^{-1}\,; h_{max} = 490.5\,m\right]$$

Problem 1-21. What path is travelled by an object in the last second of its free fall from a height of 78.5 m ?

[s = 34.4 m]

Problem 1-22. What is the initial velocity of a body making a projection vertically down from the height of 122 m if the body is to travel half its total path in the time interval of the last second of the motion?

$$\left[v_0 = 17\,ms^{-1}\right]$$

Problem 1-23. A missile is launched with an initial velocity of 1000 ms^{-1} at an elevation angle of 55^0.
Calculate the range the missile can theoretically reach and the maximum height of its theoretical trajectory.

[x = 95.7 km ; h = 34.1 km]

Problem 1-24. A particle makes a motion in a circle of radius r = 20 cm with a constant angular acceleration $\varepsilon = 2\,s^{-2}$.
Calculate the magnitudes of the tangential and the centripetal accelerations and also the total acceleration, each of these at the end of the fourth second of the motion.

$$\left[a_t = 0.4 ms^{-2},\quad a_n = 12.80 ms^{-2},\quad a = 12.81 ms^{-2}\right]$$

Problem 1-25. What is the radius of a rotating wheel if its circumference point has a three times greater velocity than a point 10 cm closer to the rotation axis?

[r = 0.15 m]

Problem 1-26. A wheel starts to rotate from rest with a constant angular acceleration $\varepsilon = 2\,s^{-2}$.
Calculate its number of revolutions in the first 15 seconds.

[N = 35.8]

Problem 1-27. A wheel starts to make a uniformly accelerated rotation. In the time interval of the first 5 seconds it makes 12.5 revolutions.
Calculate its angular velocity at the end of the fifth second.

$$\left[\omega = 10\pi\, s^{-1}\right]$$

1.2 DYNAMICS

The first law of motion:

> **every body continues in its state of rest or of uniform speed in a straight line unless it is compelled to change that state by forces acting on it.**

Notes: a) This law states that if no net force is acting on a body it remains at rest, or if it is moving it continues moving with constant speed in a straight line.
b) The tendency of a body to maintain its state of rest or of uniform motion in a straight line is called inertia; that is why the first law of motion is often called the law of inertia.

The second law of motion:
the acceleration of an object is directly proportional to the net force acting on it and it is inversely proportional to its mass.