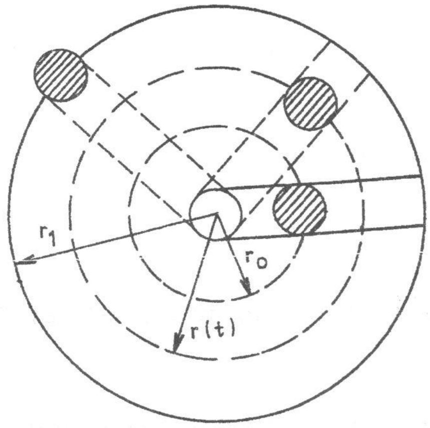
Problem 1-57. A circular disk of radius $r_1 = 1.3\ m$ rotates around the vertical axis with constant angular velocity $\omega = 8\ s^{-1}$. At a distance $r_0 = 0.5\ m$ from the axis there is a sphere of mass m = 0.25 kg inside a radial groove on the disk.



Determine: a) The time dependence of the distance of the sphere from the disk's centre.

b) The dependence of the sphere's relative velocity v' on the distance from the disk's centre.

c) The magnitude of the Coriolis' force acting on the sphere at the disk's periphery.

$[r(t) = r_0 \cosh \omega t]$

$\left[v'(r) = \omega\sqrt{r^2 - r_0^2}\right]$

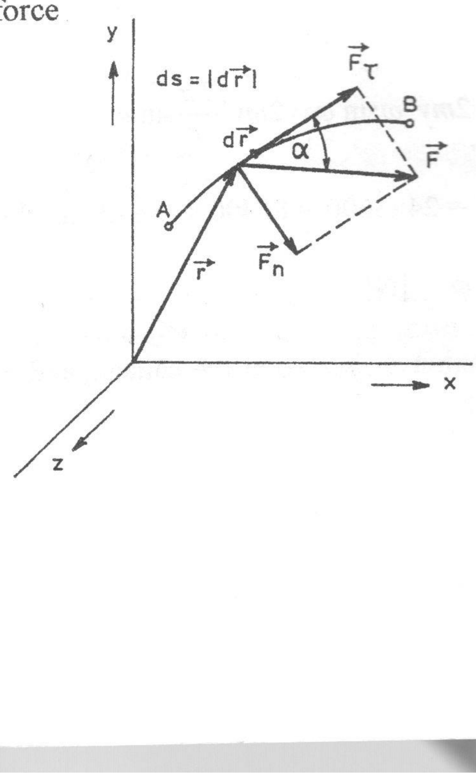
$\left[F_c(r_1) = 2m\omega^2\sqrt{r_1^2 - r_0^2} = 38.4\,N\right]$

1.4 WORK AND ENERGY

Specifically, the **work** done on a particle by a **constant force** (constant in both magnitude and direction) is defined as the product of the magnitude of the displacement times the component of the force parallel to the displacement.
In equation form this can be written as

$$W = F\ d\ \cos\varphi$$

where F is the magnitude of the constant force, *d* is the displacement of the particle, and φ is the angle between the directions of the force and the displacement (the term F $\cos\varphi$ is the component of the force parallel to the displacement).
Note that, when the force is perpendicular to the motion, no work is done by that force.



In many cases a **force varies** in magnitude or direction during a process. In this case the exact result for the work done equals

$$W = \int_A^B F\,ds\cos\varphi$$

or, using dot-product notation, we can write

$$W = \int_A^B \mathbf{F}.d\mathbf{r}$$

This is the most general definition of work.

The direction of the vector **dr** is along the tangent to the curve, φ is the angle between **F** and **dr** at any point.
Note that work is a scalar quantity.
In SI units work is measured in **newton-meters.** For convenience, a special name is given to this unit, the **joule (J): 1 J = 1 N.m**

Let us now suppose that a particle of mass m is moving in a straight line with an initial speed v_1. To accelerate it uniformly to a speed v_2, a constant force F is exerted on it parallel to its motion over distance d. Then the work done on it is $W = F\ d$.
If we apply the second law of motion we find

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

We define the quantity $\frac{1}{2}mv^2$ as the **translational kinetic energy(KE)** of the particle:

$$KE = \frac{1}{2}mv^2$$

and we can write

$$W = \Delta\ KE$$

and we see that the **net work done on an object is equal to its change in kinetic energy.** This is known as the **work-energy theorem.**
This theorem tells us that if (positive) work W is done on a body, its kinetic energy increases by an amount W. The theorem also holds true for the reverse situation: if negative work W is done on the body, the body's kinetic energy decreases by an amount W. That is, a force exerted on a body opposite to the body's direction of motion reduces its speed.
We summarise: if the net work W done on an object is positive, its KE increases, whereas if W is negative, its KE decreases. If the net work done on the object is zero, its KE remains constant.
Because of the direct connection between work and kinetic energy, **energy must be measured in the same units as work: joules in SI units.**
Like work, kinetic energy is a scalar quantity.

A force is defined as a **conservative force** if the work done by the force on a particle moving between any two positions depends only on the initial and final positions and so is independent of the particular path taken. An equivalent definition of a conservative force is also as follows: a force is a conservative force if the work done by the force is zero whenever a particle moves along any closed path that returns it to its original position.
For example, the force of gravity is conservative, and the elastic force (F=kx) is also conservative. However, the force of friction, for example, must be a nonconservative force.

Kinetic energy is associated with a moving particle. Now we introduce **potential energy(PE),** which is the energy associated with the position or configuration of a body and its surroundings. Potential energy can be defined only in relation to a conservative force.

Potential energy for mechanical systems is, like KE, closely related to the concept of work. The most common example of potential energy is **gravitational potential energy.** We know that the change in gravitational potential energy is defined so that it is equal to the negative of the work done by gravity when the object moves from height y_1 *to* y_2 :

$$\Delta U = -W_g = -\int_1^2 \mathbf{F_g}.d\mathbf{r}$$

There are other types of potential energy besides gravitational. In general we define **the change in potential energy** associated with a particular conservative force **F** as the negative of the work done by that force:

$$\Delta U = U_2 - U_1 = -\int_1^2 \mathbf{F}.d\mathbf{r} \qquad (I)$$

This definition cannot be used to define potential energy for all possible forces. It makes sense only for conservative forces. The last integral can be evaluated only if **F** is a function of position, and the integral has a unique value (which equals the work done) only if it depends only on the end points (1 and 2) and not on the path taken.
We summarise: potential energy is always associated with a conservative force, and the difference in potential energy between two points is defined by the integral (I). The choice of where the PE=0 is arbitrary and can be chosen for convenience. Since a force is always exerted on one body by another body (the earth exerts a gravitational force on a falling body; a compressed spring exerts a force on a body, and so on), potential energy is not something a body has by itself, but rather is associated with the interaction of two (or more) bodies.
In the one-dimensional case the potential energy can be written

$$U(x) = -\int F(x)dx$$

This relation tells us how to obtain U(x) when given F(x). If we are given U(x) we can obtain F(x) by inverting the above equation:

$$F(x) = -\frac{dU(x)}{dx}$$

In three dimensions, this relation becomes

$$\mathbf{F}(x,y,z) = -\mathbf{i}\frac{dU}{dx} - \mathbf{j}\frac{dU}{dy} - \mathbf{k}\frac{dU}{dz}$$

According to the work-energy theorem, the net work W done on a particle is equal to the change in KE of the particle:

$$W = \Delta KE$$

Since we have a conservative system, we can write the net work done in terms of potential energy

$$\Delta U = -\int_1^2 \mathbf{F}.d\mathbf{r} = -W$$

Thus we can write

$$\Delta KE + \Delta U = 0$$

and we can say that if the kinetic energy of the system increases, the potential energy decreases by an equal amount, and vice versa.
We now define the quantity E, called the **total mechanical energy** of our system, as the sum of the kinetic energy KE and the potential energy U:

$$E = KE + U$$

Thus, for a conservative system we have

$$E = KE + U = \text{constant}$$

and we see that the **total mechanical energy of a conservative system remains constant**, or we can also say that the **total mechanical energy is conserved.** This is called the **principle of conservation of mechanical energy** for conservative forces (the conservation of mechanical energy is an example of a more general law, the law of conservation of energy, which includes all forms of energy).

Power is defined as the rate at which work is done. **Instantaneous power**, P, is

$$P = \frac{dW}{dt}$$

Average power, $\overline{P}$, when an amount of work W is done in a time t is

$$\overline{P} = \frac{W}{t}$$

In SI units, power is measured in **joules per second** and this is given a special name, the **watt (W): 1 W** = 1 Js^{-1}.
It is often convenient to write the power in terms of the net force **F** applied to an object and its velocity **v**. Since P = dW/dt and dW = **F** . **dr** , then

$$P = \frac{dW}{dt} = \mathbf{F} . \frac{d\mathbf{r}}{dt}$$

$$P = \mathbf{F} . \mathbf{v}$$

Problem 1-58. A particle is moving along an elliptical path whose parametric equations are x(t) = a cos kt and y(t) = b sin kt .
Calculate the work done by the force causing this motion in the time between instants t=0 and t = π/4k .

<u>Solution</u>: From the definition of work we write

$$W = \int \mathbf{F}. d\mathbf{r} = \int \left(F_x dx + F_y dy \right)$$

The components of the force can be expressed from the second law of motion:

$$F_x = m\frac{d^2x}{dt^2} = -mak^2\cos kt$$

$$F_y = m\frac{d^2y}{dt^2} = -mbk^2\sin kt$$

The components of the displacement vector **dr**(dx,dy) are given as follows:

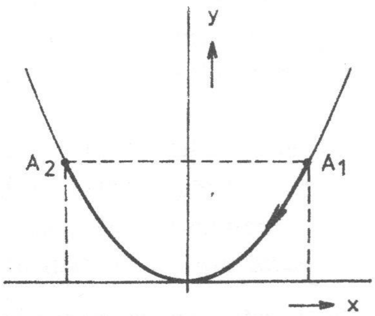
$$dx = \frac{dx}{dt}dt = -ak\sin kt\, dt$$

$$dy = \frac{dy}{dt}dt = bk\cos kt\, dt$$

Now, we can calculate the integral

$$W = \int_{t=0}^{\frac{\pi}{4k}} \left(F_x dx + F_y dy\right) = \frac{mk^2}{4}\left(a^2 - b^2\right)$$

Problem 1-59. Calculate the work done by the force $\mathbf{F} = \left(x^2 - 2xy\right)\mathbf{i} + \left(y^2 - 2xy\right)\mathbf{j}$ along the path given by parametric equations $x = t$, $y = t^2$ (parabola) from the point $A_1(1,1)$ to the point $A_2(-1,1)$.



Solution: From the definition we write

$$W = \int_{A_1}^{A_2} \mathbf{F}.d\mathbf{r} = \int_{A_1}^{A_2}\left(F_x dx + F_y dy\right) =$$

$$= \int_{A_1}^{A_2}\left[\left(x^2 - 2xy\right)dx + \left(y^2 - 2xy\right)dy\right]$$

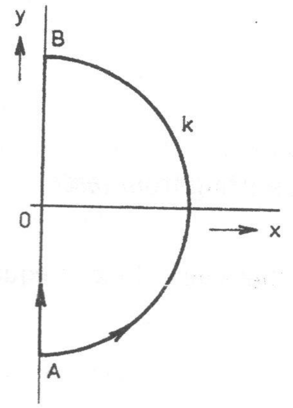
To calculate this line integral we express it in terms of parameter t and we must transform its limits.

From parametric equations we have $t = \frac{y}{x}$ and thus the parameter t for the lower limit has the value $t_1 = 1$ and for the upper limit it has the value $t_2 = -1$.

Thus, the work done equals

$$W = \int_{t_1=1}^{t_2=-1} \left[\left(t^2 - 2t^3\right)dt + \left(t^4 - 2t^3\right)2t\, dt\right] = \frac{14}{15}J$$

Problem 1-60. A body moves from the point A(0,-1,0) to the point B(0,1,0) owing to the force $\mathbf{F} = 2y^2\mathbf{i} + x^2\mathbf{j} + 3y\mathbf{k}$ acting on it. The first time it moves along the y-axis. The second time it moves in a circle in the XY plane. The centre of the circle is placed in the origin of the co-ordinate system.



Calculate the work done by a given force for each case.

Solution: a) the work done by **F** if the body moves along the y-axis equals

$$W = \int_A^B \mathbf{F}.d\mathbf{r} = \int_A^B F_y dy = 0$$

since the F_y component along the y-axis is zero.

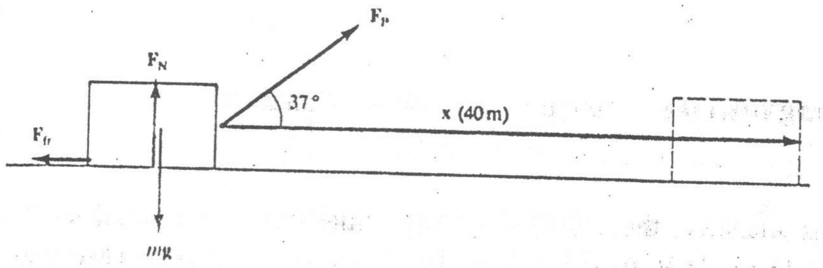
b) the work done by **F** if the body moves in a given circle equals

$$W = \int_A^B \mathbf{F}.d\mathbf{r} = \int_A^B F_x dx + \int_A^B F_y dy = \int_0^0 F_x dx + \int_{-1}^1 F_y dy = \int_{-1}^1 F_y dy = \int_{-1}^1 x^2 dy = \int_{-1}^1 (1-y^2) dy = \frac{4}{3} J$$

Problem 1-61. A 50-kg box is pulled 40 m along a horizontal floor by a constant force F_p = 100 N which acts at a 37⁰ angle. The floor is rough and exerts a friction force $F_{fr} = 50\ N$.

Determine the work done by each force acting on the box, and the net work done on the box.

Solution:



There are four forces acting on the box: $\mathbf{F}_p$, the friction force $\mathbf{F}_{fr}$, the box's weight **mg**, and the normal force $\mathbf{F}_N$ exerted upward by the floor.

The work done by the gravitational and normal forces is zero, since they are perpendicular to the displacement.

The work done by $\mathbf{F}_p$ is

$$W_p = F_p x \cos 37^0 = 3200\,J.$$

The work done by the friction force is

$$W_{fr} = F_{fr} x \cos 180^0 = -2000\,J$$

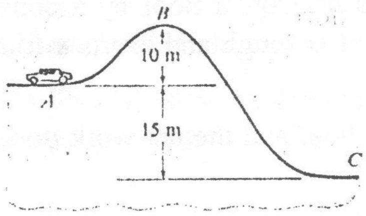
Finally, the net work done on the box is the sum of the work done by each force:

$$W = 3200\ J - 2000\ J = 1200\ J$$

Problem 1-62. A 145-g baseball is thrown with a speed of 25 m/s .
What is its KE? How much work was done to reach this speed starting from rest?

Solution: KE = $\frac{1}{2}mv^2 = 45\,J$. Since the initial KE was zero, the work done is equal to the final KE, 45 J.

Problem 1-63. A 1000-kg car moves from point A to point B and then to point C (as shown in Fig.).
a) What is the gravitational PE of the car-earth system when the car is at point B and at point C relative to point A as the zero of PE?
b) What is the change in potential energy when the car goes from B to C?
c) Repeat parts (a) and (b) but take the reference point to be at point C.



Solution: a) Let us measure the heights from point A, which means that initially the PE is zero. At point B, where y = 10 m,

$$PE_B = mgy = 1.0\times10^5\ J.$$

At point C, y = -15 m, since C is below A. Therefore,

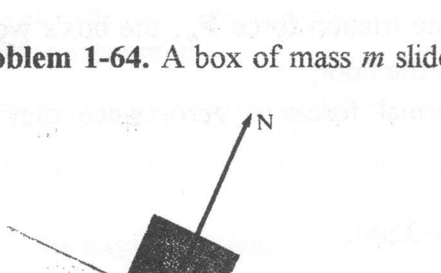
$$PE_C = mgy = -1.5\times10^5\ J.$$

b) In going from B to C, the potential energy change is

$$PE_C - PE_B = -2.5\times10^5\ J.$$

c) In this instance, the potential energy initially (at A) is equal to PE = $1.5\times10^5\ J$, since y = 15 m. At B, the PE is $2.5\times10^5\ J$, and at C it is zero. However, the change in PE going from B to C is the same as in part (b).

Problem 1-64. A box of mass m slides down a frictionless plane inclined at angle Θ from the horizontal. It begins at a height h above the bottom of the incline moving down with speed v_0.
Find the work done by all the forces and the speed of the box when it reaches the bottom.



Solution: The forces acting on the box are the force of gravity **mg** and the normal force **N** exerted by the plane. The force **N** is perpendicular to the plane and to the motion of the box. It therefore does no work on the box. The only force that does work is mg, which has the component mg sinΘ in the direction of motion. When the box moves a distance Δs down the incline, the force of gravity does work mg sinΘ Δs. Since this force is constant, the total work done when the box moves a distance *s* down the incline is merely mg sinΘ s. Since the total distance *s* measured along the incline is related to the initial height *h* by h = s .sinΘ , the work done by the force of gravity is mg sinΘ s = mgh. And since this is the total work done by all the forces, the work-energy theorem gives

$$W_{net} = mgh = \Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

The velocity at the bottom of the incline is thus given by

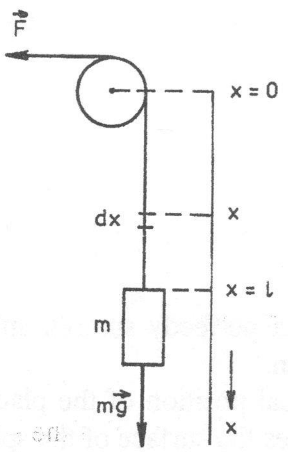
$$v = \sqrt{v_0^2 + 2gh}$$

This result is the same as if we dropped the box from a distance *h* above the surface of the earth with initial downward speed v_0. The work done by the force of gravity on the box is mgh, independent of the angle of the incline. If the angle Θ were increased, the box would travel a smaller distance *s* to drop the same vertical distance *h*, but the component of mg parallel to the motion mg sinΘ would be greater, making the work done the same.

Problem 1-65. Calculate the work required to lift a body of mass *m* when a massless pulley is used. The rope length is L.

a) calculate for the case when you disregard the rope's mass

b) calculate for the case if the mass of 1m of rope equals m_0



Solution: a) An easy case: W = F.L = mgL

b) We introduce the vertical axis x with positive orientation directed down. The mass of any rope element whose distance from the pulley is x and whose length is dy equals $m_0 dx$ and its weight equals $m_0 g dx$.

The work required to lift such a rope element equals $dW' = m_0 g x dx$

and the total work required to lift the whole rope equals

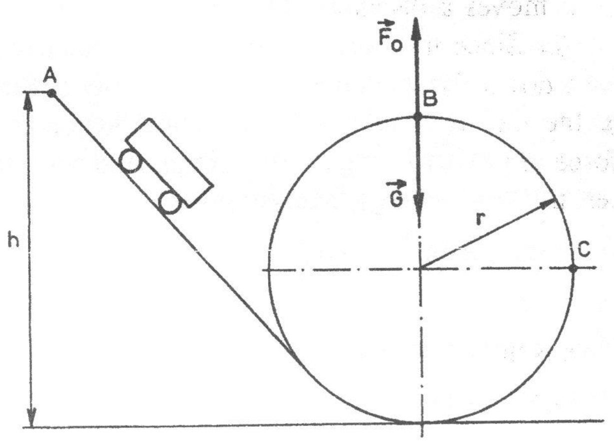
$$W' = \int_0^L dW' = m_0 g \int_0^L x dx = \frac{1}{2} m_0 g L^2$$

and the total work for the body to be lifted equals

$$W + W' = mgL + \frac{1}{2} m_0 g L^2 = gL\left(m + \frac{1}{2} m_0 L\right)$$

Problem 1-66. The trajectory of a pair of rails lies in the vertical plane and has the shape shown in Fig. From the point A whose height is h a cart starts to move with zero initial velocity.

State the height h for the cart to travel the whole circular trajectory. Friction is disregarded.



Solution: There are two forces acting on the cart at point B: the force of gravity G = mg and the centrifugal force $F_0 = \frac{mv_B^2}{r}$, when v_B is the velocity of the cart at B and r is the radius of the trajectory.

For the cart not to collapse at point B we must write

$$F_0 \geq G \quad or \quad \frac{mv_B^2}{r} \geq mg \quad and\ thus \quad v_B^2 \geq gr$$

Let us apply the principle of conservation of mechanical energy for points A and B :

$$E_A = E_B \quad or \quad KE_A + PE_A = KE_B + PE_B$$

and thus

$$mgh = \frac{1}{2}mv_B^2 + mg2r$$

From here, for the velocity at point B we have

$$v_B = \sqrt{2g(h-2r)}$$

and thus

$$2g(h-2r) \geq gr$$

This condition carries the result

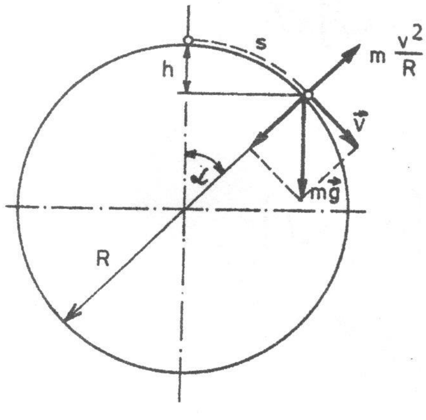
$$h \geq \frac{5}{2}r$$

Problem 1-67. A particle starts moving from the top of a perfectly smooth sphere whose radius R = 1.5 m.

Calculate: a) the vertical position of the place at which the particle leaves the surface of the sphere

b) the path travelled on the surface of the sphere

c) the velocity at the instant of leaving the surface of the sphere



Solution: At the leaving point the radial component of the force of gravity must be equal to the centrifugal force:

$$mg\cos\alpha = m\frac{v^2}{R}$$

where $\cos\alpha = \frac{R-h}{R}$. Then $g(R-h) = v^2$.

The principle of conservation of mechanical energy gives us

$$mgh = \frac{1}{2}mv^2 \quad or \quad v^2 = 2gh$$

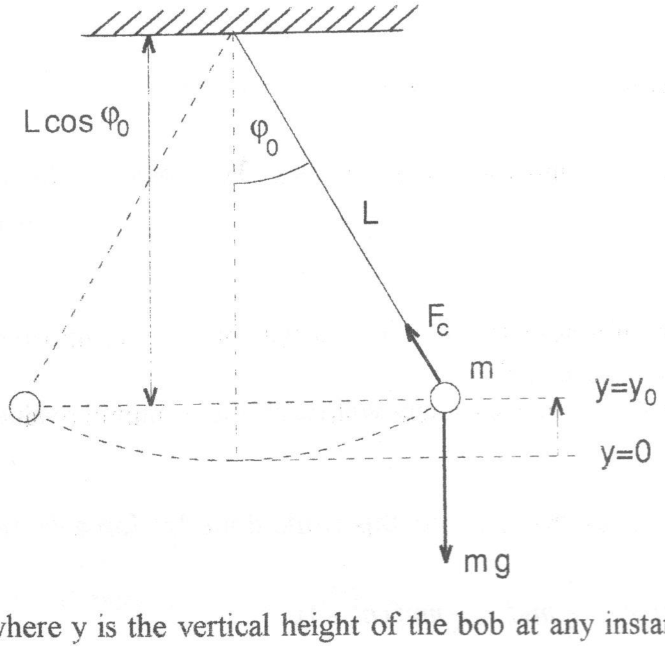
Then $\quad g(R-h) = 2gh \quad$ or $\quad h = \frac{R}{3} = 0.5m$

and $\quad v = \sqrt{2gh} = 3.13ms^{-1}$.

As for the path travelled on the surface of the sphere

$$s = R\alpha = R\arccos\frac{2}{3} = 1.2m$$

Problem 1-68. A simple pendulum of mass m is released at t = 0 when the cord makes an angle $\varphi = \varphi_0$ to the vertical. The length of the massless cord is L.
a) Determine the speed as a function of position and its magnitude at the lowest point.
b) Determine the tension in the cord.



Solution: Two forces are acting on the bob at any moment: the force of gravity mg and the force F_C that the cord exerts on the bob.
We put the y-axis in the vertical direction and the origin y=0 is taken at the lowest point of the bob's path.
We apply the principle of conservation of mechanical energy. The total mechanical energy of the bob is

$$E = \frac{1}{2}mv^2 + mgy$$

where y is the vertical height of the bob at any instant and v is its speed at the same instant.
At t = 0 the y-coordinate equals $\quad y = y_0 = L - L\cos\varphi_0 = L(1-\cos\varphi_0)$
and the total energy equals $\quad E = mgy_0$.

At any other instant $$E = \frac{1}{2}mv^2 + mgy = mgy_0$$

Thus $$v = \sqrt{2g(y_0 - y)}$$

This is the dependence of v on the vertical coordinate y.

And also $$v = \sqrt{2gL(\cos\varphi - \cos\varphi_0)}$$

This is the dependence of v on the angular displacement φ.

For the lowest point $y = 0$ and $\varphi = 0$, so

$$v = \sqrt{2gy_0} \quad or \quad v = \sqrt{2gL(1 - \cos\varphi_0)}$$

These expressions represent the maximum speed.

The tension in the cord is the force F_C that the cord exerts on the bob. The bob moves in a circle and there must be the radial acceleration $\frac{v^2}{L}$ and the centripetal force $m\frac{v^2}{L}$ must be equal to the net force acting on the bob in the direction of the cord.

This net force in the cord direction equals force F_C minus the component of gravity mg $\cos\varphi$ that acts outward. So,

$$m\frac{v^2}{L} = F_C - mg\cos\varphi$$

Then $$F_C = m\left(\frac{v^2}{L} + g\cos\varphi\right)$$

Using the result for the function $v(\varphi)$

$$F_C = m[2g(\cos\varphi - \cos\varphi_0) + g\cos\varphi] = mg(3\cos\varphi - 2\cos\varphi_0)$$

Note that for $\varphi = \varphi_0$ $F_C = mg\cos\varphi_0$ and for $\varphi = 0$ $F_C = mg(3 - 2\cos\varphi_0)$ - this is the maximum value of F_C.

Problem 1-69. The initial velocity of a particle is v_0. The particle has a constant force F exerted on it. The mass of the particle equals m.

Calculate the trajectory on which the particle's velocity will reach the n-multiple value of the initial velocity.

Solution: The increase in its KE must be equal to the work done by force F on trajectory x:

$$F x = \frac{1}{2}mn^2v_0^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}mv_0^2(n^2 - 1)$$

Thus $$x = \frac{mv_0^2(n^2 - 1)}{2F}$$

Problem 1-70. A body is making a free fall. What is its velocity at height h if its velocity at height $h_0 \rangle h$ *was* v_0. (air resistance is disregarded)

Solution: From the principle of conservation of energy

$$\frac{1}{2}mv_0^2 + mgh_0 = \frac{1}{2}mv^2 + mgh$$

$$v = \sqrt{v_0^2 + 2g(h_0 - h)}$$

Problem 1-71. A force of 500 N is acting on a certain body. What is the work done by the force if the body moves a distance of 60 m

a) In the direction of the force?
b) At a 60^0 angle to the force?
c) At right angles to the force?
d) In a direction opposite to that of the force?

[30 000 J, 15 000 J, 0, -30 000 J]

Problem 1-72. A force of 1N is applied to a 500-g particle which is initially at rest on a smooth horizontal surface. If the force is tangential to the surface, how much work does it do on the particle in 10 s?

[W = 0.1 kJ]

Problem 1-73. At what rate must energy be expended to raise 1000 kg/min of water to a height of 22 m if the water is discharged from the top of the pipe at a speed of 4m/s ?

[P = 3.7 kW]

Problem 1-74. A train with a mass of $2 \times 10^5\ kg$ is travelling at a uniform speed of 65 km/h. What is its kinetic energy in joules and in kilowatt-hours ?

$[3.3 \times 10^7\ J]$

$[9.2\ kWh]$

Problem 1-75. A constant horizontal force of magnitude F acts on an object of mass m which is initially at rest on a smooth horizontal surface. Find the instantaneous power P expended by the force at any time t.

$$\left[P(t) = \frac{F^2 t}{m}\right]$$

Problem 1-76. The top of an incline is 1 m higher than the bottom. If a 100-g body sliding down this incline acquires a speed of 180m/min, how much work has been done against friction ?

[0.53 J]

Problem 1-77. A car of 1500-kg mass moves with velocity $v_0 = 5.5 ms^{-1}$. Calculate the force that will increase its velocity to the magnitude v = 8.57 m/s along the path s = 75 m.

[F = 432 N]

Problem 1-78. A train traveling with a velocity $v_1 = 68\, km\, h^{-1}$ is able to stop on the path $s_1 = 400 m$. What is its velocity v_2 for which it can be stopped along a path $s_2 = 100 m$?

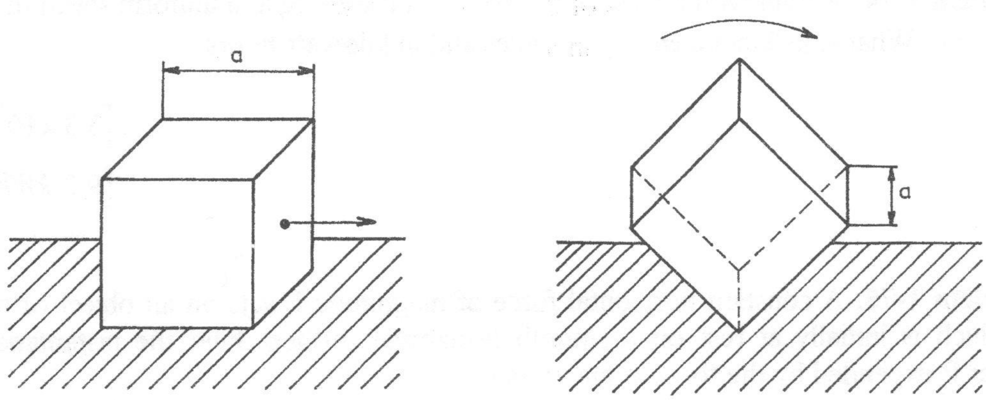
[v_2 = 34 km/h]

Problem 1-79. What is the work required to turn over a cube whose edge has magnitude a and whose mass density equals ρ ?

$$\left[W = \frac{a^4 \rho g}{2}\left(\sqrt{2}-1\right) \right]$$

Problem 1-80. We want to displace a cube each side of which equals a to a distance d First the cube is pulled along the ground, and then it is continually turned over its edge. For what friction coefficient is the work done to displace the cube the same for the both ways of transferring it ? (see Fig.)

$$\left[\mu = \frac{1}{2}\left(\sqrt{2}-1\right) \right]$$



Problem 1-81. A body of 1-kg mass projected vertically upward has at a height of 10 m a KE of 50 J.
Calculate the velocity of its fall to the earth's surface.

[v = 17.2 m/s]

Problem 1-82. What is the average power of a crane that heaves a $10^4 - kg$ load to a height of 6 m in a time of 2 minutes ?

$$\left[\bar{P} = 4.9\, kW\right]$$

Problem 1-83. A train of $10^6 - kg$ mass starts to move from rest and in a time of 60 s it will reach the velocity of 50 km/h.
What is the work done by the engine and what is its average power ?

$$\left[W = 96 \times 10^6\, J, \quad \bar{P} = 1600\, kW\right]$$

Problem 1-84. A lift and its load have a total mass m = 1200 kg. To travel a distance of 20 m the lift needs a time of 4 s.
What is the work done in this time and what is the average power of the lift's engine ?

$$\left[W = 235.4\, kJ, \quad \bar{P} = 58.85\, kW\right]$$

1.5 SYSTEMS OF PARTICLES

We consider a system of particles of total mass

$$M = m_1 + m_2 + \ldots = \sum_i m_i$$

where m_i is the mass of the i-th particle.
Let $\mathbf{r}_i$ be the position vector for the i-th particle relative to some arbitrary origin. The vector $\mathbf{r}_{cm}$ which is defined by

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$$

represents the location of the **center of mass** of the system.
By differentiating each term with respect to time we obtain the **velocity of the center of mass**

$$\mathbf{v}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{v}_i$$

The quantity $m_i \mathbf{v}_i$ is called the **linear momentum** of the i-th particle.