Problem 1-81. A body of 1-kg mass projected vertically upward has at a height of 10 m a KE of 50 J.
Calculate the velocity of its fall to the earth's surface.

[v= 17.2 m/s]

Problem 1-82. What is the average power of a crane that heaves a $10^4 - kg$ load to a height of 6 m in a time of 2 minutes ?

$$\left[\overline{P} = 4.9\,kW\right]$$

Problem 1-83. A train of $10^6 - kg$ mass starts to move from rest and in a time of 60 s it will reach the velocity of 50 km/h.
What is the work done by the engine and what is its average power ?

$$\left[W = 96\times 10^6\,J,\quad \overline{P} = 1600\,kW\right]$$

Problem 1-84. A lift and its load have a total mass m = 1200 kg. To travel a distance of 20 m the lift needs a time of 4 s.
What is the work done in this time and what is the average power of the lift's engine ?

$$\left[W = 235.4\,kJ,\quad \overline{P} = 58.85\,kW\right]$$

1.5 SYSTEMS OF PARTICLES

We consider a system of particles of total mass

$$M = m_1 + m_2 + \ldots = \sum_i m_i$$

where m_i is the mass of the i-th particle.
Let $\mathbf{r}_i$ be the position vector for the i-th particle relative to some arbitrary origin. The vector $\mathbf{r}_{cm}$ which is defined by

$$\mathbf{r}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$$

represents the location of the **center of mass** of the system.
By differentiating each term with respect to time we obtain the **velocity of the center of mass**

$$\mathbf{v}_{cm} = \frac{1}{M}\sum_i m_i \mathbf{v}_i$$

The quantity $m_i\mathbf{v}_i$ is called the **linear momentum** of the i-th particle.

If we rewrite this relation

$$M\mathbf{v}_{cm} = \sum_i m_i \mathbf{v}_i$$

the right side of the equation is the total moment of the particles in the system and we have the important result: **the total moment of a system of particles equals the product of the total mass M and the velocity of the center of mass $\mathbf{v}_{cm}$**

$$\mathbf{P} = \sum_i m_i \mathbf{v}_i = M\mathbf{v}_{cm}$$

By again differentiating each term with respect to time we find the **acceleration of the center of mass**

$$M\mathbf{a}_{cm} = \sum_i m_i \mathbf{a}_i$$

According to the second law of motion, the mass of each particle times its acceleration equals the resultant force $\mathbf{F}_i$ acting on the particle

$$M\mathbf{a}_{cm} = \sum_i \mathbf{F}_i$$

This equation states that the **center of mass moves like a particle of mass** $M = \sum_i m_i$ **under the influence of the resultant external force on the system.**

We see that the center of mass of the system behaves just like a simple particle, subject only to external forces.

An important special case of the motion of a system of particles occurs when the resultant external force is zero. In this case we have

$$M\mathbf{a}_{cm} = M\frac{d\mathbf{v}_{cm}}{dt} = 0$$

and

$$M\mathbf{v}_{cm} = \mathbf{P} = \sum_i m_i \mathbf{v}_i = \text{constant}$$

This important result is called the **law of conservation of linear momentum.** It states that **if the resultant external force on a system is zero, the velocity of the center of mass of the system is constant and the total linear momentum of the system is constant (is conserved).**

If the center of mass of a continuous body is to be calculated, the sum must be replaced by the integral

$$\mathbf{r}_{cm} = \frac{1}{M}\int \mathbf{r}\, dm$$

where dm is an element of mass and **r** is its position vector.

The center of mass coordinates of a continuous body are then calculated

$$x_{cm} = \frac{1}{M}\int x\, dm$$

$$y_{cm} = \frac{1}{M}\int y\, dm$$

$$z_{cm} = \frac{1}{M}\int z\, dm$$

Problem 1-85. Examine the relative velocity of two particles after an elastic collision. Assume that the particles have masses m_1 *and* m_2 and they move along the x axis and their velocities before the elastic collision are v_1 *and* v_2, respectively.
Determine also their velocities v_1', v_2' for some special cases:

a) $m_1 = m_2$

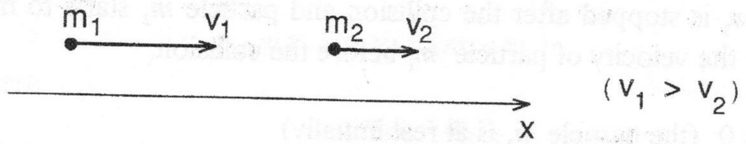
b) $m_1 \neq m_2$ *and* $v_2 = 0$

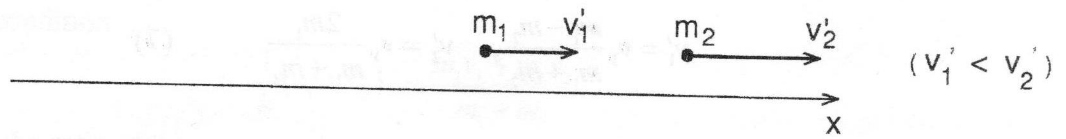
c) $m_1 = m_2$ *and* $v_2 = 0$

d) $m_1 \gg m_2$ *and* $v_2 = 0$

e) $m_1 \ll m_2$ *and* $v_2 = 0$

<u>Solution</u>:





We make the assumption that the velocity of each particle will be considered to be positive if moving to the right (increasing x) and the velocity will be considered to be negative if moving to the left (decreasing x).

The law of conservation of linear momentum yields

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2' \qquad (1)$$

For an elastic collision the total kinetic energy is also conserved

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \qquad (2)$$

By solving these two equations we have

$$v_1 - v_2 = v_2' - v_1' \qquad (3)$$

This result tells us that for any head-on collision the relative velocity of two particles after the collision is the same as that existing before, no matter what the masses are.

a) $m_1 = m_2$
From Eq.(1) we have

$$v_1 + v_2 = v_1' + v_2' \qquad (4)$$

If we add and subtract Eqs.(3) and (4) then

$$v_1' = v_2 \quad and \quad v_2' = v_1 \qquad (5)$$

The particles exchange their velocities as a result of the collision.

If particle m_2 is at rest before the collision, so that $v_2 = 0$, then

$$v_1' = 0 \quad and \quad v_2' = v_1 \qquad (6)$$

that is, particle m_1 is stopped after the collision and particle m_2 starts to move with a velocity equal to the velocity of particle m_1 before the collision.

b) $m_1 \neq m_2$, $v_2 = 0$ (the particle m_2 is at rest initially)
In this case we have the result

$$v_1' = v_1 \frac{m_1 - m_2}{m_1 + m_2}, \quad v_2' = v_1 \frac{2m_1}{m_1 + m_2} \qquad (7)$$

c) $m_1 = m_2$, $v_2 = 0$
From Eqs.(7) we have

$$v_1' = 0 \quad and \quad v_2' = v_1$$

This is the same as result (6).

d) $m_1 \gg m_2$, $v_2 = 0$ (a very heavy moving particle strikes a light object at rest)
From Eqs.(7) we have the result

$$v_1' \approx v_1$$
$$v_2' \approx 2v_1$$

The velocity of the heavy particle is practically unchanged, while the light particle, originally at rest, takes twice the velocity of the heavy particle.

e) $m_1 \ll m_2$, $v_2 = 0$ (a moving light particle strikes a very heavy particle at rest)
From Eqs.(7) we have the result

$$v_1' \approx -v_1$$
$$v_2' \approx 0$$

The heavy particle remains at rest and the very light particle rebounds with its same velocity but in the opposite direction.

Problem 1-86. A carriage of mass m_1 going with velocity v_1 strikes another one of mass m_2 going in the same direction with velocity $v_2 \langle v_1$. After the collision they join together.

Calculate the velocity v of their motion after the collision and state the ratio $\frac{m_1}{m_2}$ for velocity v to be described by the relation $v = \frac{1}{k}(v_1 + v_2)$, where k is a given number larger 1.

Solution: Write the law of conservation of linear momentum

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)v$$

and from here

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

From the condition

$$\frac{1}{k}(v_1 + v_2) = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

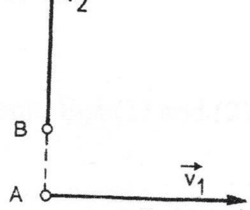
we get for the ratio of the masses

$$\frac{m_1}{m_2} = \frac{v_2(k-1) - v_1}{v_2 - (k-1)v_1}$$

Problem 1-87. We consider a system of two particles A and B of masses m_1 *and* m_2, respectively, which attract each other. Initially, particle B has velocity $\mathbf{v}_2$ in the direction of their connecting line and particle A has velocity $\mathbf{v}_1$ perpendicular to this connecting line (see Fig.).

Calculate the velocity of the center of mass of the system.

Solution: Since no external forces act on the system we may apply the law of conservation of linear momentum



$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = \text{constant}$$

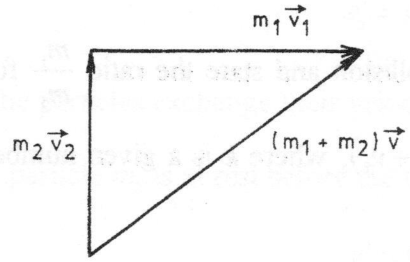
The total momentum of the system is also equal to the linear momentum of the center of mass of the system. If $\mathbf{v}$ is the velocity of the centre of mass then we may write

$$(m_1 + m_2)\mathbf{v} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2$$

and from here

$$\mathrm{v} = \frac{\mathrm{m_1 v_1 + m_2 v_2}}{\mathrm{m_1 + m_2}}$$

Graphically, the result can be expressed as shown in Fig. and therefore we can write



$$(m_1 + m_2)^2 v^2 = m_1^2 v_1^2 + m_2^2 v_2^2$$

or

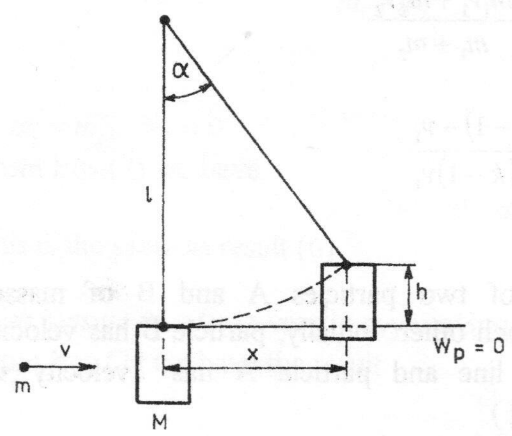
$$v = \sqrt{\frac{m_1^2 v_1^2 + m_2^2 v_2^2}{(m_1 + m_2)^2}}$$

Problem 1-88. Calculate the velocity of a projectile from the displacement of a ballistic pendulum if you know the mass m of the projectile, the mass M of the ballistic pendulum and its length L.

Solution: We apply the law of conservation of linear momentum and the law of conservation of energy

$$mv = (m + M)V$$

$$\frac{1}{2}(m+M)V^2 = (m+M)gh$$



when v is the velocity of the projectile to be calculated and V is the initial velocity of the system (projectile plus pendulum) after loading the projectile into the pendulum.

Since

$$h = L(1 - \cos\alpha) = 2L\sin^2\frac{\alpha}{2}$$

then

$$v = \frac{m+M}{m} 2\sqrt{gL} \sin\frac{\alpha}{2}$$

If we measure distance x instead of angle α, then

$$h = L - \sqrt{L^2 - x^2}$$

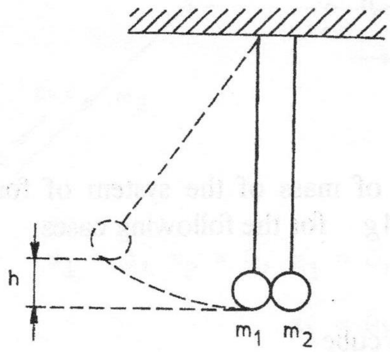
In practice $x << L$, so $h \approx \frac{x^2}{2L}$

and thus

$$v = \frac{m+M}{m} x \sqrt{\frac{g}{L}}$$

Problem 1-89. Two spheres of masses m_1 *and* m_2, *when* $m_1 = 2m_2$, are suspended at the same height. Initially, the spheres touch each other. At one moment the heavier sphere is displaced to a height h and released.
What heights will the spheres reach after their elastic collision ?

Solution: We introduce: v - velocity of sphere m_1 before the collision
v_1 - velocity of sphere m_1 after the collision
v_2 - velocity of sphere m_2 after the collision



We write the law of conservation of linear momentum of the system

$$m_1 v = m_1 v_1 + \frac{m_1}{2} v_2$$

and thus

$$v = v_1 + \frac{v_2}{2} \quad (1)$$

The law of conservation of energy gives us

$$\frac{1}{2} m_1 v^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{4} m_1 v_2^2$$

and thus

$$v^2 = v_1^2 + \frac{v_2^2}{2}$$

Thus we have

$$\left(v_1 + \frac{v_2}{2} \right)^2 = v_1^2 + \frac{v_2^2}{2}$$

and from here

$$v_1 = \frac{v_2}{4} \quad (2)$$

Velocity v can be calculated from the law of mechanical energy for sphere m_1

$$m_1 g h = \frac{1}{2} m_1 v^2$$

and thus

$$v^2 = 2gh \quad (3)$$

From Eqs.(1) and (2) we can calculate

$$v_1 = \frac{1}{3} v, \qquad v_2 = \frac{4}{3} v \quad (4)$$

Now we can determine the heights of both spheres after their collision :

$$m_1 g h_1 = \frac{1}{2} m_1 v_1^2 \quad ; \quad m_2 g h_2 = \frac{1}{2} m_2 v_2^2$$

$$g h_1 = \frac{1}{2}\frac{1}{9} v^2 \quad ; \quad g h_2 = \frac{1}{2}\frac{16}{9} v^2$$

$$g h_1 = \frac{1}{18} 2gh \quad ; \quad g h_2 = \frac{16}{18} 2gh$$

$$h_1 = \frac{1}{9} h \quad ; \quad h_2 = \frac{16}{9} h$$

Problem 1-90. Calculate the location of the centre of mass of the system of four particles of masses $m_1 = 1g,\ m_2 = 2g,\ m_3 = 3g,\ m_4 = 4g$ for the following cases:

a) the particles lie on a straight line

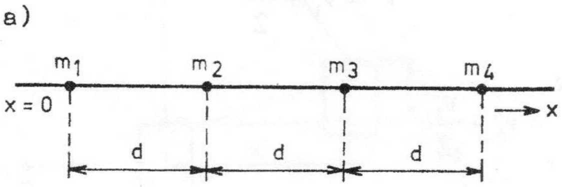
b) the particles lie at the apexes of a square

c) the particles lie at the four neighbouring apexes of a cube

The distances between neighbouring particles are equal to d = 10 cm for all configurations.

Solution:

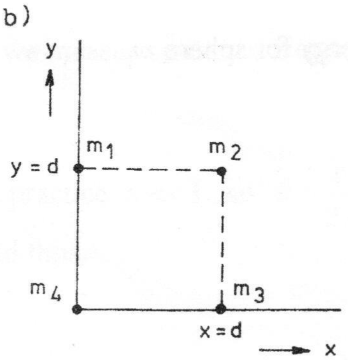
a)



$$x_o = \frac{\sum_{i=1}^{4} x_i m_i}{\sum_{i=1}^{4} m_i} = 2d = 20 \text{ cm}$$

$x_1 = 0,\ x_2 = d,\ x_3 = 2d,\ x_4 = 3d$ $\qquad y_o = 0\ ,\ z_o = 0\ .$

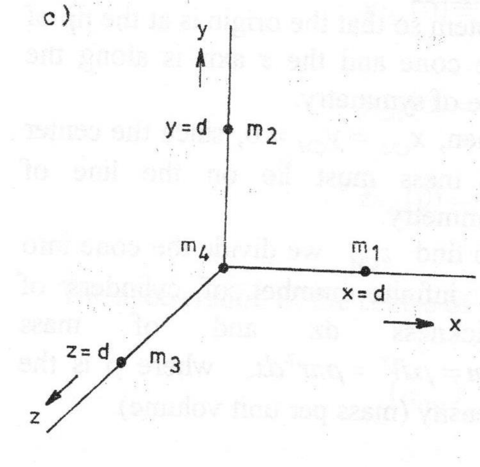
b)



$$x_o = \frac{\sum_{i=1}^{4} x_i m_i}{\sum_{i=1}^{4} m_i} = \frac{x_2 m_2 + x_3 m_3}{\sum_{i=1}^{4} m_i} = \frac{2d + 3d}{10} = \frac{d}{2} = 5 \text{ cm}$$

$$y_o = \frac{\sum_{i=1}^{4} y_i m_i}{\sum_{i=1}^{4} m_i} = \frac{y_1 m_1 + y_2 m_2}{\sum_{i=1}^{4} m_i} = \frac{d + 2d}{10} = \frac{3}{10} d = 3 \text{ cm}$$

$x_1 = 0,\ x_2 = d,\ x_3 = d,\ x_4 = 0$ $\qquad y_1 = d,\ y_2 = d,\ y_3 = 0,\ y_4 = 0$



$$x_o = \frac{\sum_{i=1}^{4} x_i m_i}{\sum_{i=1}^{4} m_i} = \frac{x_1 m_1}{10} = \frac{d}{10} = 1\ \text{cm},$$

$$y_o = \frac{\sum_{i=1}^{4} y_i m_i}{\sum_{i=1}^{4} m_i} = \frac{y_2 m_2}{10} = \frac{2d}{10} = 2\ \text{cm},$$

$$z_o = \frac{\sum_{i=1}^{4} z_i m_i}{\sum_{i=1}^{4} m_i} = \frac{z_3 m_3}{10} = \frac{3d}{10} = 3\ \text{cm}.$$

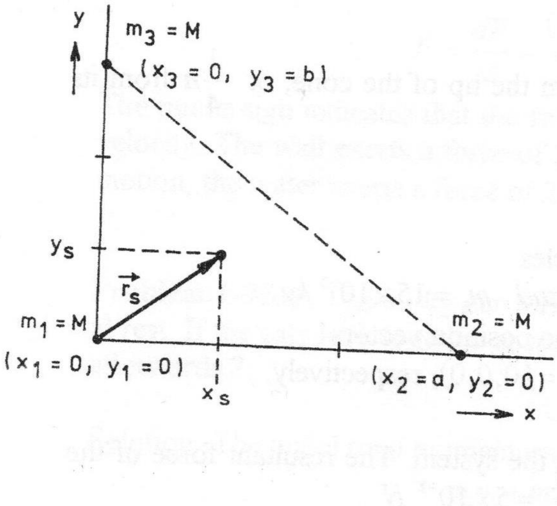
$$x_1 = d,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0, \qquad y_1 = 0,\ y_2 = d,\ y_3 = 0,\ y_4 = 0,$$

$$z_1 = 0,\ z_2 = 0,\ z_3 = d,\ z_4 = 0.$$

Problem 1-91. Three particles m_1, m_2 *and* m_3, whose masses have the same magnitude M, are located at the apexes of a right-angled triangle the legs of which are equal to a, b.

Determine the centre of mass of the system of particles.

Solution: The total mass of the system equals m = 3 M.



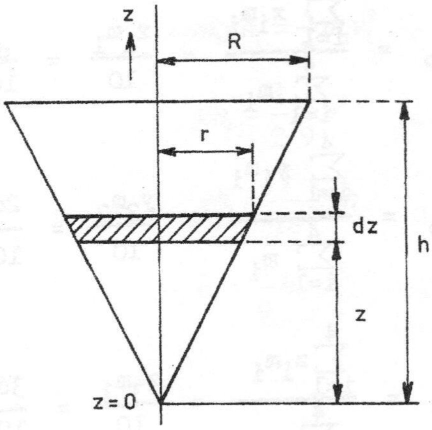
$$x_s = \frac{1}{m}\sum_{i=1}^{3} m_i x_i =$$

$$= \frac{1}{3M}(m_1 x_1 + m_2 x_2 + m_3 x_3) = \frac{1}{3M} Ma = \frac{a}{3};$$

$$y_s = \frac{1}{m}\sum_{i=1}^{3} m_i y_i =$$

$$= \frac{1}{3M}(m_1 y_1 + m_2 y_2 + m_3 y_3) = \frac{1}{3M} Mb = \frac{b}{3}.$$

Problem 1-92. Determine the center of mass of a uniform cone of height h and radius R.



Solution: We choose the coordinate system so that the origin is at the tip of the cone and the z axis is along the line of symmetry.

Then, $x_{CM} = y_{CM} = 0$, since the center of mass must lie on the line of symmetry.

To find z_{CM} we divide the cone into an infinite number of cylinders of thickness dz and of mass $dm = \rho dV = \rho\pi r^2 dz$, where ρ is the density (mass per unit volume).

As the volume of such an infinitesimal cylinder is $dV = \pi r^2 dz$ we write

$$z_{CM} = \frac{1}{M}\int z\,dm = \frac{1}{M}\int_0^h z\rho\pi r^2 dz$$

The radius of each infinitesimal cylinder can be expressed from the ratio r/z=R/h, so r=Rz/h . Now we have

$$z_{CM} = \frac{1}{M}\int z\rho\pi\frac{R^2 z^2}{h^2}dz = \frac{\rho\pi R^2}{Mh^2}\int_0^h z^3 dz = \frac{\rho\pi R^2 h^2}{4M}$$

The total mass M of the cone equals the density ρ times the total volume of the cone $V = \frac{1}{3}\pi R^2 h$, that is, $M = \frac{1}{3}\rho\pi R^2 h$. Thus we have the result

$$z_{CM} = \frac{3}{4}h$$

Thus, the center of mass of the cone is $\frac{3}{4}h$ from the tip of the cone, or $\frac{1}{4}h$ from its base.

Problem 1-93. Consider a system of three particles

$$m_1 = 5\times 10^{-3}\,kg,\ m_2 = 10^{-2}\,kg \ \ and \ \ m_3 = 15\times 10^{-3}\,kg.$$

At time t = 0 their locations are described by the position vectors

$$\mathbf{r}_1 = (3;4,5),\ \mathbf{r}_2 = (-2;4;-6),\ \mathbf{r}_3 = (0;0;0), \text{ respectively.}$$

All coordinates are given in centimeters.

At time t = 0 the external forces start to act on the system. The resultant force of the acting forces is equal to $\mathbf{F} = (F_x;0;0)$ *where* $F_x = 5\times 10^{-2}\ N$.

Calculate the location of the centre of mass of the system at time t = 2 s.

Solution: For the coordinates of the centre of mass at time t = 0 we have

$$x_{CM}(0) = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = -0.17\ cm$$

$$y_{CM}(0) = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = 2\ cm$$

$$z_{CM}(0) = \frac{m_1z_1 + m_2z_2 + m_3z_3}{m_1 + m_2 + m_3} = -1.17\ cm$$

The acceleration of the centre of mass is equal

$$(a_{CM})_x = \frac{F_x}{m_1 + m_2 + m_3} = 1.67\ ms^{-2}$$

Thus, the location of the centre of mass at time t = 2 s is given by the coordinates

$$x_{CM}(2) = x_{CM}(0) + \frac{1}{2}(a_{CM})_x t^2 = 333.8\ cm$$

$$y_{CM}(2) = y_{CM}(0)$$

$$z_{CM}(2) = z_{CM}(0)$$

Problem 1-94. Water leaves a hose at a rate of 5 kg/s with a speed of 50 m/s. It strikes a wall, which stops it.
What is the force exerted by the water on the wall ?

Solution: In each second, water with a momentum of $5 \times 50 = 250$ kg.m/s is brought to rest. The magnitude of the force that the wall exerts on the water to change the momentum by this amount is

$$F = \frac{dP}{dt} = \frac{(0-250)kg.m/s}{1s} = -250\,N$$

The minus sign indicates that the force exerted on the water is opposite to its original velocity. The wall exerts a force of 250 N to stop the water. Thus, by the third law of motion, the water exerts a force of 250 N on the wall.

Problem 1-95. A 10,000-kg car traveling at a speed of 24 m/s strikes an identical car at rest. If the cars lock together as a result of the collision, what is their common speed afterwards ?

Solution: The initial total momentum is

$$m_1v_1 + m_2v_2 = 24 \times 10^4\ kg.m/s$$

After the collision, the total momentum will be the same but will be shared by both cars. Since the two cars have become attached, they will have the same velocity, call it v'.
Then:

$$(m_1 + m_2)v' = 24 \times 10^4 \, kg.ms^{-1}$$
$$v' = 12 ms^{-1}$$

Problem 1-96. A homogeneous plank lies on a perfectly smooth horizontal plane. A man walks along the plank with constant relative speed u.
Calculate the absolute speeds v_1 *and* v_2 of the man and the plank, respectively. The mass of the man is m_1 and the mass of the plank is m_2.

$$\left[v_2 = -\frac{m_1 u}{m_1 + m_2} \right]$$

$$\left[v_1 = u + v_2 = \frac{m_2 u}{m_1 + m_2} \right]$$

Problem 1-97. Two vessels sail on still water towards to each other. When passing they exchange a heavy bag of M = 50-kg mass. As a result, one vessel stops and the other continues moving in its original direction with speed $u_1 = 8.5 ms^{-1}$.
Calculate the speeds v_1 *and* v_2 of the vessels before exchanging the bags. The masses of the vessels inclusive of the bag are equal $m_1 = 1000 kg$, $m_2 = 500 kg$.

$$\left[v_2 = -\frac{m_1 u_1 M}{m_1 m_2 - M(m_1 + m_2)} = -1 ms^{-1} \right]$$

$$\left[v_1 = \frac{m_1 u_1 - m_2 u_2}{m_1} = 9 ms^{-1} \right]$$

Problem 1-98. A hunter stands on a very smooth ice plane. What is an initial speed of his motion after he fires off his rifle. The mass of the hunter inclusive of the rifle equals m = 75 kg and the mass of the projectile is $m_1 = 10 g$. The initial speed of the projectile is $v_1 = 750 ms^{-1}$.

$$\left[v = \frac{m_1 v_1}{m} = 0.1 ms^{-1} \right]$$

Problem 1-99. A vessel of m_1 = 100-kg mass and d = 2.5 m long is at rest on water. A man of m = 70-kg mass walks from the tail to the bow of the vessel.
What is the displacement of the vessel ? (water resistance is disregarded)

$$\left[x = \frac{m}{m + m_1} d = 1.03 \, m \right]$$