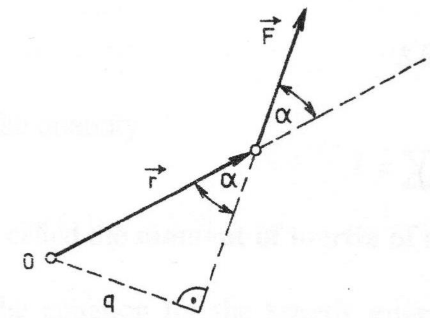
1.6 ROTATIONAL DYNAMICS



Firstly, we define the moment of a force. If a force **F** acts on a simple particle at a point whose position with respect to the origin 0 of the inertial reference frame is given by the position vector **r**, the **torque** τ acting on the particle **with respect to the origin 0** is defined as the vector product

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

Torque is a vector quantity and its magnitude is given by

$$\tau = r\,F \sin\alpha$$

where α is the angle between **r** and **F**; its direction is normal to the plane formed by **r** and **F**. Its orientation is given by the right-hand rule for the vector product of two vectors. Torque has the dimensions $kg\,m^2\,s^{-2}$ and the **unit of torque** is the newton-meter (N.m).

Note that the torque depends not only on the magnitude and on the direction of the force but also on the point of application of the force relative to the origin 0, that is, on vector **r**.

We can also write the magnitude of vector τ either as

$$\tau = (r \sin\alpha)F = F\,r_{\perp}$$

or as

$$\tau = r(F \sin\alpha) = r\,F_{\perp}$$

in which $r_{\perp}$ is the component of **r** at right angles to the line of action of **F**, and $F_{\perp}$ is the component of **F** at right angles to **r**.

Torque is often called the **moment of force** and $r_{\perp}$ is called the **moment arm**.

We see that only the component of **F** perpendicular to **r** contributes to the torque. In particular, when α equals 0^0 *or* 180^0, there is no perpendicular component; then the line of action of the force passes through the origin and the moment arm $r_{\perp}$ about the origin is also zero and the torque τ is zero.

The analog of linear momentum in rotation motion is an angular momentum. Consider a particle of mass *m* and linear momentum **p** at a position **r** relative to the origin 0 of an inertial reference frame. We define the **angular momentum L** of the particle **with respect to the origin 0** as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

Note that we must specify the origin 0 in order to define the position vector **r** in the definition of angular momentum.

Angular momentum is a vector. Its magnitude is given by

$$L = r\,p \sin\varphi$$

where φ is the angle between **r** and **p**; its direction is normal to the plane formed by **r** and **p** and its orientation is given by the right-hand rule.

Angular momentum has the dimensions $kg\,m^2\,s^{-1}$.

We now derive an important relation between torque and angular momentum. We have seen that $\mathbf{F} = d(m\mathbf{v})/dt = d\mathbf{p}/dt$ for a particle. We take the vector product of $\mathbf{r}$ with both sides of this equation:

$$\mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

But $\mathbf{r} \times \mathbf{F}$ is the torque about 0. We can then write

$$\tau = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad (1)$$

Next we differentiate the Eq. for $\mathbf{L}$

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$

or

$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

We rewrite the right side of this equation

$$\frac{d\mathbf{L}}{dt} = (\mathbf{v} \times m\mathbf{v}) + \mathbf{r}\frac{d\mathbf{p}}{dt}$$

Now $\mathbf{v} \times m\mathbf{v} = 0$, because the vector product of two parallel vectors is zero. Therefore,

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \frac{d\mathbf{p}}{dt} \qquad (2)$$

Inspection of Eqs.(1) and (2) shows that

$$\tau = \frac{d\mathbf{L}}{dt} \qquad (3)$$

which states that **the time rate of change of the angular momentum of a particle is equal to the torque acting on it.**

This result is the rotational analog of the equation $\mathbf{F} = d\mathbf{p}/dt$ which states that the time rate of change of the linear momentum of a particle is equal to the force acting on it.

Any moving body may be considered as a system of individual particles, each with kinetic energy $\frac{1}{2}mv^2$. The kinetic energy of the body itself is equal to the sum of the kinetic energies of its particles

$$KE = \frac{1}{2}\left(m_1 v_1^2 + m_2 v_2^2 + \ldots\right)$$

If the motion is pure translation, so that all particles have the same linear speed v, this equation reduces to $KE = \frac{1}{2}Mv^2$, where M is the total mass of the body.

If the motion is pure rotation, the linear speeds v_i of the particles differ according to the distances R_i of the particle from the axis of rotation. However, the angular speeds ω are the same, provided the body is rigid, and since $v_i = R_i\omega$ for pure rotation, we can express the kinetic energy of a rotating body as

$$KE = \frac{1}{2} = \sum m_i R_i^2 \omega^2$$

The quantity

$$I = \sum m_i R_i^2 \qquad [I] = kg.m^2$$

is called the **moment of inertia** of the body about the axis of rotation.

The equation for the **kinetic energy of a rotating rigid body** then reduces to the form

$$KE = \frac{1}{2} I\omega^2$$

Note the similarity of this expression to that for kinetic energy of translation, $KE = \frac{1}{2}Mv^2$; the quantity I corresponds to mass M and the angular speed ω corresponds to linear speed v.

It is evident from the definition that I may be found by imagining the rotating body divided into minute particles, multiplying the mass of each particle by the square of its distance from the axis, and then adding these products. It is important to note that R_i does not refer to the position vector of m_i; it is the perpendicular distance from m_i to the axis of rotation. Hence I depends on the position as well as the direction of the axis.

Just as the inertia or mass of a body is a measure of the resistance it offers to linear acceleration, the moment of inertia I is a measure of its resistance to angular acceleration. However, it is evident that I is not proportional to mass alone; the moment of inertia is a function both of the total mass and of the distribution of mass - that is, of the distance of the mass elements dm from the axis of rotation. Therefore, in the case of continuous bodies, it is generally necessary to calculate I with the aid of integral calculus. For this reason, the summation for I becomes an integral as the masses m_i become infinitesimal mass elements dm.

So, in the case of a continuous body, the moment of inertia is calculated as

$$I = \int_0^M R^2 dm = \int_V R^2 \rho dV$$

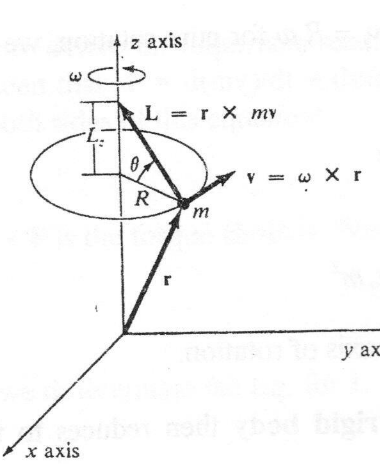
where M is the total mass of the body, R is the perpendicular distance of dm from the axis of rotation, and ρ is the density of the body, which may be a function of position.

For purely rotational motion of a particle m at a distance R from a fixed axis of rotation, the angular momentum component along that axis is

$$L_z = L \sin \Theta = mvr \sin \Theta$$

However, r sinΘ = R and v = R ω, so that

$$L_z = mR^2 \omega$$



For a rigid body, then, in which all particles m_i rotate with the same angular velocity ω about an axis of rotation that is fixed with respect to the body, the angular momentum of the body along the axis of rotation is

$$L = \sum_i m_i R_i^2 \omega = I\omega$$

where the moment of inertia I is computed with respect to the axis of rotation.

Eq.(3) now yields

$$\tau = \frac{d(I\omega)}{dt}$$

In the case of constant I, the torque can also be expressed as

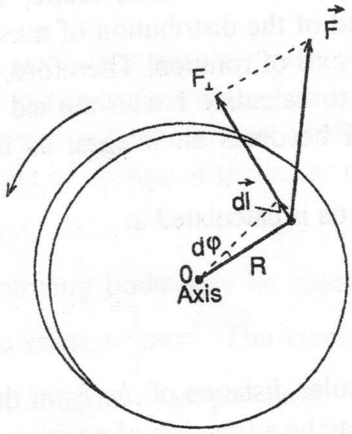
$$\tau = I\frac{d\omega}{dt} = I\varepsilon$$

This is the rotational analog of F = m a. It represents the second law of motion for rotation.

When there is no torque, angular momentum **L** must be constant over time

$$\frac{dL}{dt} = 0 \quad \textit{and then} \quad L = I\omega = \text{constant}$$

This is the **law of conservation of angular momentum** for a rotating body: **for a rigid body, the angular momentum as well as the angular velocity remains constant if the sum of the externally applied torques is zero.**



Let us now calculate the work done on a body rotating about a fixed axis. We suppose that a force **F** acts at a point at a perpendicular distance **R** from the axis of rotation.

The work done by this force is

$$W = \int \mathbf{F}.d\mathbf{l} = \int F_{\perp} R\, d\varphi$$

But $F_{\perp} R$ is the torque about the axis, so

$$W = \int \tau\, d\varphi$$

The work-energy theorem must also hold for the rotation of a rigid body about a fixed axis.

As we can write

$$\tau = I\varepsilon = I\frac{d\omega}{dt} = I\frac{d\omega}{d\varphi}\frac{d\varphi}{dt} = I\omega\frac{d\omega}{d\varphi}$$

$$W = \int_{\varphi_1}^{\varphi_2} \tau d\varphi = \int_{\omega_1}^{\omega_2} I\omega d\omega = \frac{1}{2} I\omega_2^2 - \frac{1}{2} I\omega_1^2$$

The work-energy theorem for a body rotating about a fixed axis states that the work done in a rotating body through an angle $\varphi_2 - \varphi_1$ is equal to the change in rotational kinetic energy of the body.

Power P is given by

$$P = \frac{dW}{dt} = \tau \frac{d\varphi}{dt} = \tau\omega$$

Note that this is the rotational analog of $P = F.v$

The parallel-axis theorem states that if I is the moment of inertia of a body of total mass M about any axis, and I_{CM} is the moment of inertia about an axis passing through the center of mass and parallel to the first axis at distance h away, then

$$I = I_{CM} + Mh^2$$

The **equilibrium condition** for uniform translation motion of the center of mass is

$$\sum_i \mathbf{F}_i = 0$$

and the **equilibrium condition** for steady rotation about some axis is

$$\sum_i \tau_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = 0$$

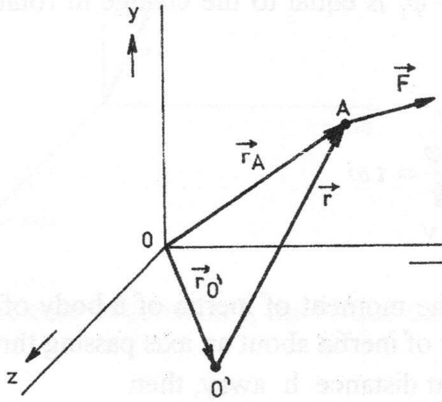
where $\mathbf{r}_i$ specifies the point of application of $\mathbf{F}_i$ relative to a selected origin.

Combined translations and rotations: If a body is not mounted on a fixed axis, but is free to move in space, the applied forces will generally produce translational motion of the center of mass, as well as rotational motion about some axis through the center of mass.

In this case, the total kinetic energy of a moving body is the sum of its kinetic energy of rotation about the center of mass and its kinetic energy of translation associated with the motion of the center of mass

$$KE = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} Mv_{CM}^2$$

Problem 1-100. The force $\mathbf{F} = 3\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}$ (SI units) acts at the point $A = [5;7;6]$. State the magnitude and the direction of the torque vector considered with respect to the point $0' = [3;-5;2]$.



Solution: The position vector of the point of application A with respect to the reference point 0′ equals

$$\mathbf{r} = \mathbf{r}_A - \mathbf{r}_{0'} = 2\mathbf{i} + 12\mathbf{j} + 4\mathbf{k}$$

thus, the torque with respect to point 0′ equals

$$\mathbf{M} = \mathbf{r} \times \mathbf{F} = 20\mathbf{i} + 4\mathbf{j} - 22\mathbf{k}$$

and the magnitude of **M** equals

$$M = \sqrt{400 + 16 + 484} = 30\, N.m$$

The direction of **M** is given in terms of its directional cosines. These are given by the components of the unit vector in the direction of **M**.

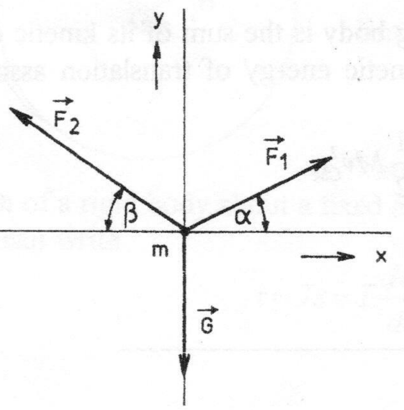
$$\mathbf{M}^0 = \frac{\mathbf{M}}{M} = \frac{2}{3}\mathbf{i} + \frac{2}{15}\mathbf{j} - \frac{11}{15}\mathbf{k}$$

and thus

$$\cos\alpha = \frac{2}{3}, \quad \cos\beta = \frac{2}{15}, \quad \cos\gamma = -\frac{11}{15}$$

Problem 1-101. A lamp of mass m is suspended on two wires. The first wire makes an angle α with the horizontal plane and the other one makes an angle β with the same plane.
Calculate the tension in the wires.

Solution:



$$\mathbf{G} = -mg\,\mathbf{j}$$

$$\mathbf{F}_1 = F_1(\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha)$$

$$\mathbf{F}_2 = F_2(-\mathbf{i}\cos\beta + \mathbf{j}\sin\beta)$$

From the equilibrium condition

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{G} = 0$$

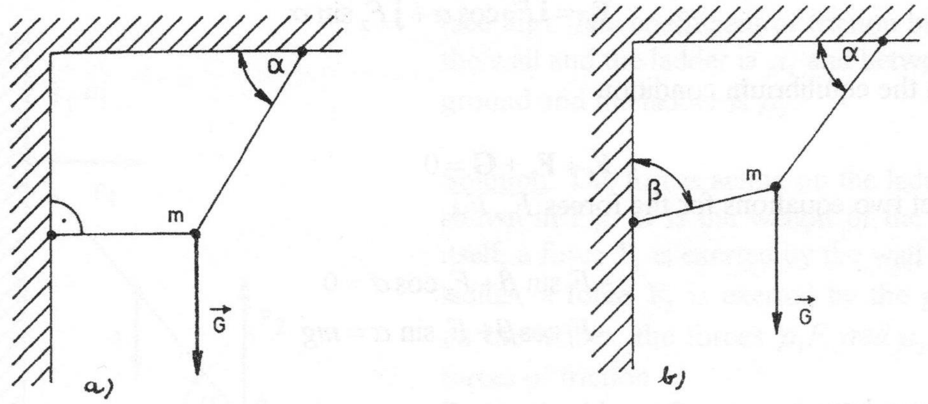
we have two equations for the tension forces in the wires

$$F_1 \cos\alpha - F_2 \cos\beta = 0$$
$$F_1 \sin\alpha + F_2 \sin\beta = mg$$

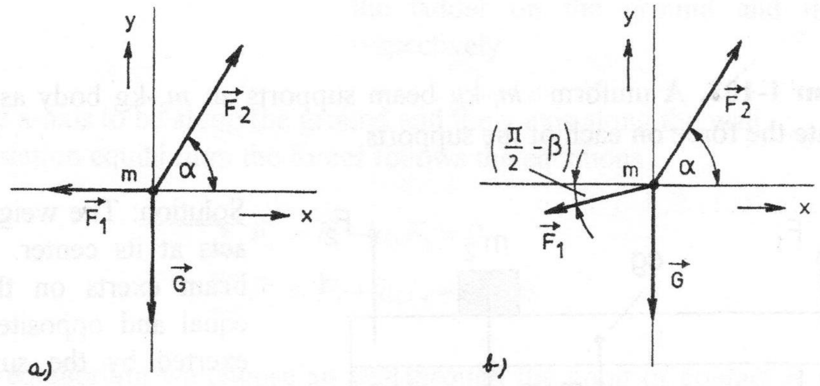
and from here

$$F_1 = mg\frac{\cos\beta}{\sin(\alpha+\beta)}; \qquad F_2 = mg\frac{\cos\alpha}{\sin(\alpha+\beta)}$$

Problem 1-102. A body of mass m is suspended as shown in Figs. Calculate the tension in the ropes for each configuration.



Solution:



a) $\mathbf{G} = -\mathbf{j}\, mg$
$\mathbf{F}_1 = -\mathbf{i}F_1$
$\mathbf{F}_2 = \mathbf{i}\,F_2 \cos\alpha + \mathbf{j}\,F_2 \sin\alpha$

From the equilibrium condition

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{G} = 0$$

we get two equations for the forces F_1, F_2;

$$-F_1 + F_2 \cos\alpha = 0$$
$$F_2 \sin\alpha - mg = 0$$

and from here

$$F_1 = \frac{mg}{tg\alpha}; \quad F_2 = \frac{mg}{\sin\alpha}$$

b) $$\mathbf{G} = -\mathbf{j}\,mg$$
$$\mathbf{F}_1 = -\mathbf{i}F_1 \cos\left(\frac{\pi}{2} - \beta\right) - \mathbf{j}F_1 \sin\left(\frac{\pi}{2} - \beta\right)$$
$$\mathbf{F}_2 = \mathbf{i}F_2 \cos\alpha + \mathbf{j}F_2 \sin\alpha$$

From the equilibrium condition

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{G} = 0$$

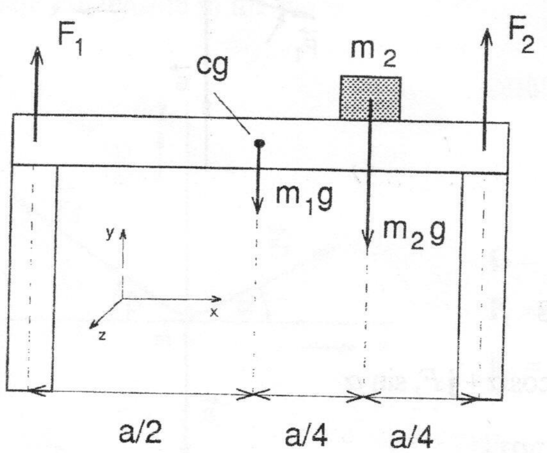
we get two equations for the forces F_1, F_2;

$$-F_1 \sin\beta + F_2 \cos\alpha = 0$$
$$-F_1 \cos\beta + F_2 \sin\alpha = mg$$

and from here

$$F_1 = -mg\frac{\cos\alpha}{\cos(\alpha+\beta)}; \quad F_2 = -mg\frac{\sin\beta}{\cos(\alpha+\beta)}$$

Problem 1-103. A uniform m_1-kg beam supports an m_2-kg body as shown in Fig. Calculate the force on each of the supports.



Solution: The weight of the beam acts at its center. The force the beam exerts on the supports is equal and opposite to the forces exerted by the supports on the beam.

Since it does not matter which point we choose as the axis for the torque equation, we calculate the torques about the point of application of $\mathbf{F}_1$, for example.

We can now write the equilibrium conditions for the beam

$$F_1 - m_1 g - m_2 g + F_2 = 0$$

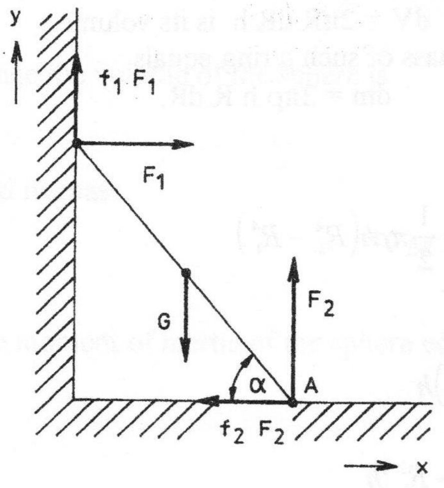
$$-m_1 g \frac{a}{2} - m_2 g \frac{3}{4} a + F_2 a = 0$$

and from here

$$F_1 = g\left(\frac{m_1}{2} + \frac{m_2}{4}\right)$$

$$F_2 = g\left(\frac{m_1}{2} + \frac{3}{4} m_2\right)$$

Problem 1-104. Calculate the minimum angle with respect to the horizontal for which the ladder does not fall due to its weight. (see Fig). The coefficient of friction between the wall and the ladder is μ_1 and between the ground and the ladder is μ_2.



Solution: The forces acting on the ladder are shown in Fig. **G** is the weight of the ladder itself, a force $\mathbf{F}_1$ is exerted by the wall on the ladder, a force $\mathbf{F}_2$ is exerted by the ground on the ladder, the forces $\mu_1 F_1$ *and* $\mu_2 F_2$ are forces of friction.

By the third law of motion the forces exerted by the ground and the wall on the ladder are equal but opposite to the forces exerted by the ladder on the ground and the wall, respectively.

We choose the x-axis to be along the ground and the y-axis along the wall.
Then, for translation equilibrium the forces follows the equations

$$\sum F_x = F_1 - \mu_2 F_2 = 0$$

$$\sum F_y = F_2 + \mu_1 F_1 - G = 0$$

For rotational equilibrium we choose an axis through the point of contact A with the ground and we obtain

$$G\frac{L}{2}\cos\alpha - \mu_1 F_1 L \cos\alpha - F_1 L \sin\alpha = 0$$

By solving these equations we get

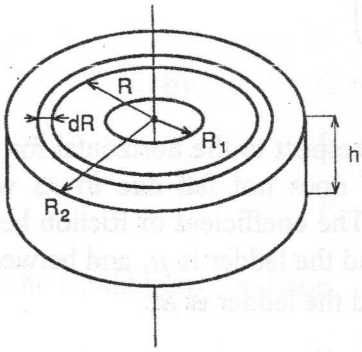
$$\frac{1}{2}\left(\mu_1 + \frac{1}{\mu_2}\right)\cos\alpha - \mu_1 \cos\alpha - \sin\alpha = 0$$

and from here we obtain the result

$$tg\alpha = \frac{1-\mu_1\mu_2}{2\mu_2}$$

The angle depends on the coefficients of friction only.

Problem 1-105. Calculate the moment of inertia of a uniform hollow cylinder of inner radius R_1, outer radius R_2 and mass M, if the rotation axis is along the axis of symmetry.



Solution: We divide the cylinder into concentric cylindrical rings of thickness dR. The mass of such a ring equals dm = ρ dV, where ρ is density and dV = 2πR.dR.h is its volume. Thus, the mass of such a ring equals

$$dm = 2\pi\rho.h.R.dR.$$

Its moment of inertia is now

$$I = \int_M R^2 dm = 2\pi\rho h \int_{R_1}^{R_2} R^3 dR = \frac{1}{2}\pi\rho h\left(R_2^4 - R_1^4\right)$$

The volume of the hollow cylinder is

$$V = \pi\left(R_2^2 - R_1^2\right)h$$

and its mass

$$M = \rho V = \rho\pi\left(R_2^2 - R_1^2\right)h$$

Thus

$$I = \frac{\pi\rho h}{2}\left(R_2^2 - R_1^2\right)\left(R_2^2 + R_1^2\right) = \frac{1}{2}M\left(R_1^2 + R_2^2\right)$$

Note that for a solid cylinder $R_1 = 0$ and if we put $R_2 = R_0$, we have a formula for the moment of inertia of a solid cylinder whose radius is R_0

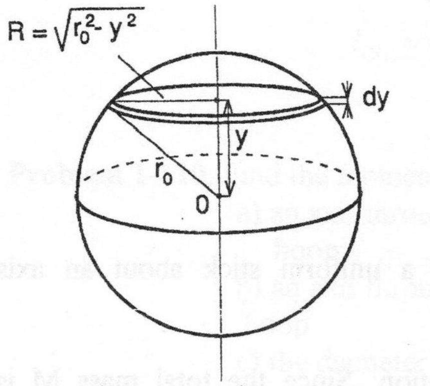
$$I = \frac{1}{2}MR_0^2$$

where M is the total mass of the solid cylinder.

Problem 1-106. Calculate the moment of inertia of a uniform solid sphere of radius r_0 and the total mass M about an axis through its center.

Solution: We divide the sphere into infinitesimal cylinders of thickness dy. Each cylinder has a radius

$$R = \sqrt{r_0^2 - y^2}$$



and its mass equals

$$dm = \rho dV = \rho\pi R^2 dy = \rho\pi\left(r_0^2 - y^2\right)dy$$

Thus, the moment of inertia of each infinitesimal cylinder is

$$dI = \frac{1}{2}dmR^2 = \frac{\pi\rho}{2}\left(r_0^4 - 2r_0^2y^2 + y^4\right)dy$$

Integrating over all these infinitesimal cylinders gives us the moment of inertia of the sphere

$$I = \frac{\rho\pi}{2}\int_{-r_0}^{r_0}\left(r_0^4 - 2r_0^2y^2 + y^4\right)dy = \frac{8}{15}\pi\rho r_0^5$$

Since the volume of the sphere is

$$V = \frac{4}{3}\pi r_0^3$$

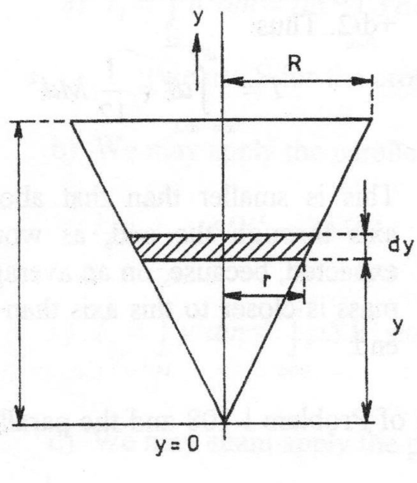
and its mass

$$M = \rho V = \frac{4}{3}\pi\rho r_0^3$$

the moment of inertia of the sphere equals

$$I = \frac{2}{5}Mr_0^2$$

Problem 1-107. Calculate the moment of inertia of a uniform cone if the rotation axis is along the axis of symmetry. (R - the radius of the cone, M - its mass)



Solution: At the distance y from the top we choose an infinitesimal cylinder whose radius is r = $\frac{R}{h}y$ and whose mass equals $dM = \pi r^2 \rho dy$.

The moment of inertia of such a cylinder equals

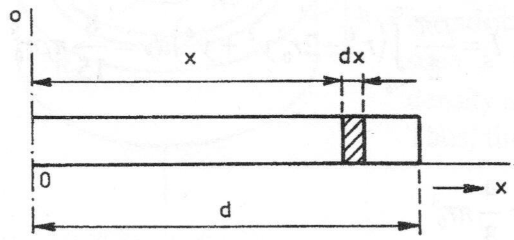
$$dI = \frac{1}{2}r^2 dM = \frac{1}{2}\pi\frac{R^4}{h^4}\rho y^4 dy$$

The moment of inertia of the cone is now calculated as the integral

$$I = \int_{y=0}^{h} dI = \frac{3}{10} MR^2$$

where $M = \frac{1}{3}\pi R^2 h\rho$ is the mass of the cone.

Problem 1-108. Find the moment of inertia of a uniform stick about an axis perpendicular to a stick through one end.



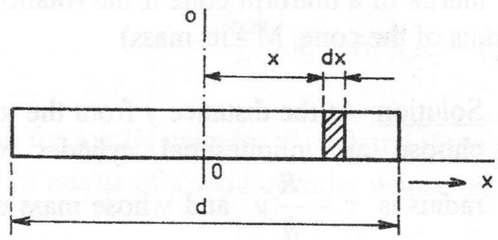
Solution: Since the total mass M is uniformly distributed along the length, the mass element dm has the moment of inertia

$$dI = x^2 dm = x^2 \rho S dx$$

and the moment of inertia of the stick is now calculated as

$$I = \int_{x=0}^{d} dI = \frac{1}{3} Md^2$$

Problem 1-109. Find the moment of inertia of a uniform stick about an axis perpendicular to a stick through the center of mass.



Solution: Since the axis is through the center of mass of the stick, the integration extends from -d/2 to +d/2. Thus

$$I = \int_{x=-d/2}^{d/2} dI = \frac{1}{12} Md^2$$

This is smaller than that about the axis through the end, as would be expected, because, on an average, the mass is closer to this axis than to the end.

This result can also be obtained from the result of Problem 1-108 and the parallel-axis theorem. In this case $h = \frac{1}{2}d$, and

$$I = I_{CM} + M\left(\frac{1}{2}d\right)^2 = \frac{1}{3} Md^2$$

or

$$I_{CM} = \frac{1}{3}Md^2 - \frac{1}{4}Md^2 = \frac{1}{12}Md^2$$

Problem 1-110. Find the moment of inertia of a hoop of mass M and radius R about

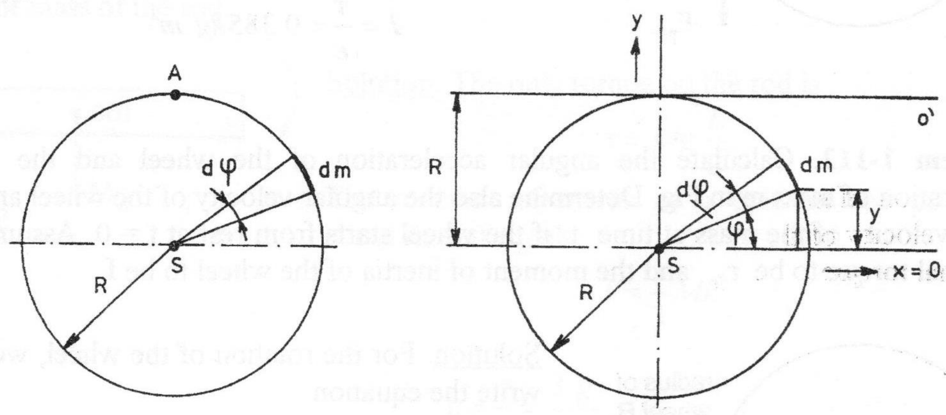
a) an axis through the center and perpendicular to the plane of the hoop

b) an axis through point A and perpendicular to the plane of the hoop

c) the diameter of the hoop

d) an axis o' that is a tangent to the hoop

Solution:



a) $I_1 = \int\limits_M R^2 dm = \rho R^2 \int\limits_{\varphi=0}^{2\pi} SRd\varphi = MR^2$

(we use S for the cross-section area)

b) We may apply the parallel-axis theorem

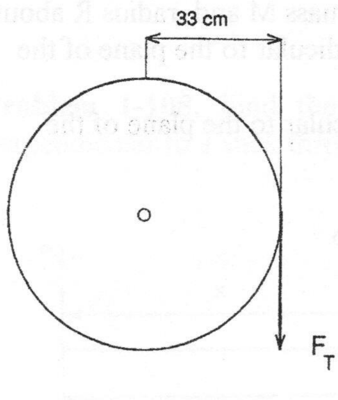
$$I_2 = I_1 + MR^2 = 2MR^2$$

c) $I_x = \int\limits_M y^2 dm = \int\limits_{\varphi=0}^{2\pi} \rho SR^3 \sin^2 \varphi d\varphi = \frac{1}{2}MR^2$

d) We may again apply the parallel-axis theorem

$$I_{o'} = I_x + MR^2 = \frac{3}{2}MR^2$$

Problem 1-111. A 15 N force F_T is applied to a cord wrapped around a wheel of radius R = 33 cm. The wheel is observed to accelerate uniformly from rest to reach an angular speed of $30s^{-1}$ in a time of 3 s. If there is a frictional torque $\tau_{fr} = 1.1N.m$, determine the moment of inertia of the wheel.



Solution: The net torque is the applied torque due to F_T minus the frictional torque

$$\tau = F_T R - \tau_{fr}$$

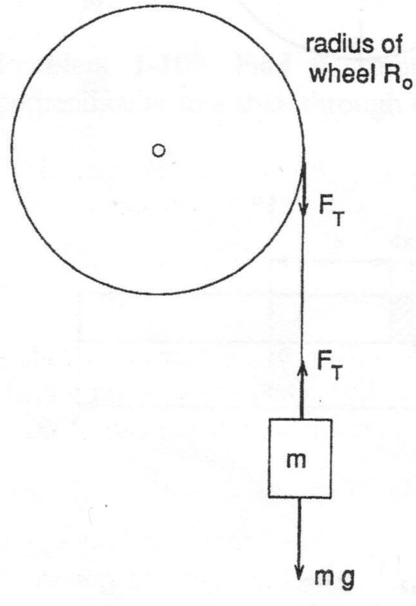
The angular acceleration is

$$\varepsilon = \frac{\Delta\omega}{\Delta t}$$

and thus

$$I = \frac{\tau}{\varepsilon} = 0.385 kg.m^2$$

Problem 1-112. Calculate the angular acceleration of the wheel and the linear acceleration of mass m in Fig. Determine also the angular velocity of the wheel and the linear velocity of the mass at time t if the wheel starts from rest at t = 0. Assume the frictional torque to be τ_{fr} and the moment of inertia of the wheel to be I.



Solution: For the rotation of the wheel, we may write the equation

$$\tau = I\,\varepsilon$$

where

$$\tau = F_T R_0 - \tau_{fr}$$

thus, the angular acceleration of the wheel equals

$$\varepsilon = \frac{\tau}{I} = \frac{F_T R_0 - \tau_{fr}}{I}$$

Next we look at the linear motion of the mass m. Two forces act on the mass: the force of gravity mg acts downward and the tension of the cord F_T upward.

Thus, we may write the equation

ma = mg - F_T

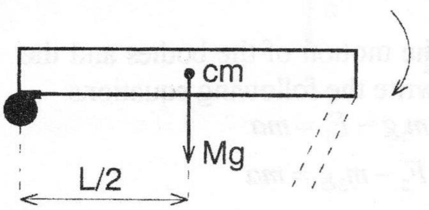
If we use the relation $a = R_0\varepsilon$, we get from last two equations

$$\varepsilon = \frac{mgR_0 - \tau_{fr}}{I + mR_0^2}$$

As all quantities on the right-hand side are constants, the angular acceleration of the wheel rotation is constant.
Thus, we can now express the linear acceleration and the linear velocity of the mass as well as the angular velocity of the wheel:

$$a = R_0 \varepsilon$$
$$\omega = \omega_0 + \varepsilon t = \varepsilon t, \quad (\omega_0 = 0 \ at \ t = 0)$$
$$v = R_0 \omega = R_0 \varepsilon t$$

Problem 1-113. A uniform rod of length L can pivot freely about a hinge as shown in Fig. The rod is released from the horizontal position. At the moment of release, determine the angular acceleration of the rod. Assume the force of gravity acts at the center of mass of the rod.



Solution: The only torque on the rod is

$$\tau = Mg\frac{L}{2}$$

The moment of inertia of a uniform rod pivoted about its end equals

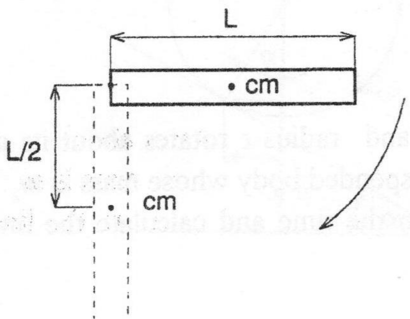
$$I = \frac{1}{3}ML^2$$

Thus

$$\varepsilon = \frac{\tau}{I} = \frac{3}{2}\frac{g}{L}$$

Note this is the angular acceleration at the moment of release. As the rod descends, the torque cannot be constant and thus the rod's angular acceleration also cannot be constant.

Problem 1-114. A rod of length L is pivoted on a frictionless hinge at one of its ends as shown in Fig. The rod is held at rest horizontally and then released. Determine the angular velocity of the rod when it reaches the vertical position, and the speed of the rod's tip at this moment.



Solution: We can use the work-energy theorem here; the work done is due to gravity. The work done by gravity is, of course, equal to the change in gravitational potential energy of the rod. Since the center of mass of the rod drops a vertical distance L/2, the work done by gravity equals

$$W = Mg\frac{L}{2}$$

The initial KE is zero. Hence, from the work-energy theorem we write

$$\frac{1}{2}I\omega^2 = Mg\frac{L}{2}$$

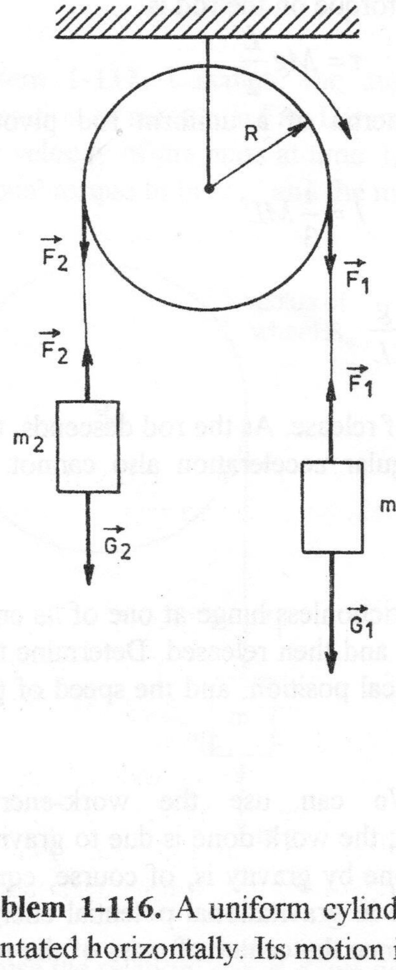
Since $I = \frac{1}{3}ML^2$ for a rod pivoted about its end, we can solve for ω

$$\omega = \sqrt{\frac{3g}{L}}$$

The tip of the rod will have the linear speed

$$v = L\omega = \sqrt{3gL}$$

Problem 1-115. Calculate the acceleration of the bodies $m_1 \rangle m_2$ suspended on a pulley whose radius is R and whose moment of inertia is I.



Solution: For the motion of the bodies and the pulley we can write the following equations:

$$m_1 g - F_1 = ma$$

$$F_2 - m_2 g = ma$$

$$(F_1 - F_2)R = \frac{a}{R}I$$

These are three equations for three unknown quantities a, F_1, F_2.

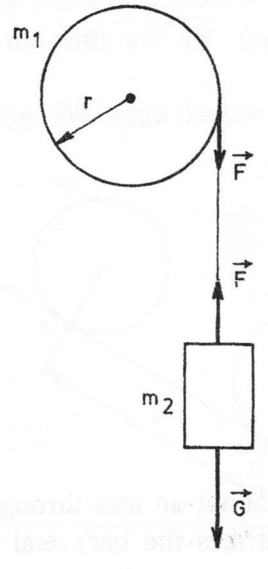
The solution for the acceleration gives the result

$$a = g\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}$$

Problem 1-116. A uniform cylinder of mass m_1 and radius r rotates about its axis orientated horizontally. Its motion is caused by a suspended body whose mass is m_2. Find the dependence of the angle displacement on the time and calculate the linear acceleration of the motion of the body m_2.

Solution: The rotational motion of the cylinder is described by the equation

$$F.r = I\frac{d^2\varphi}{dt^2} \quad (1)$$



The motion of body m_2. is described by the equation

$$G - F = m_2 a \qquad or \qquad m_2 g - F = m_2 a \qquad (2)$$

If we substitute for F from (2) to (1) and we use $I = \frac{1}{2}m_1 r^2$ and $a = r\varepsilon = r\frac{d^2\varphi}{dt^2}$

we obtain the differential equation

$$\frac{d^2\varphi}{dt^2} = \frac{2m_2 g}{(m_1 + 2m_2)r}$$

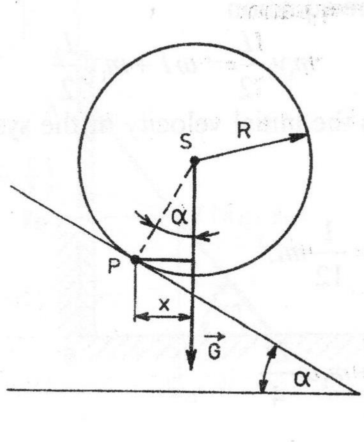
Integrating gives us (provided $\omega = 0$ and $\varphi = 0$ at time t = 0)

$$\varphi(t) = \frac{m_2 g}{(m_1 + 2m_2)r}t^2$$

Finally, body m_2 makes a uniformly accelerated motion with an acceleration

$$a = r\frac{d^2\varphi}{dt^2} = \frac{2m_2}{m_1 + 2m_2}g$$

Problem 1-117. Consider a sphere rolling down an inclined plane without slipping as in Fig. Let m and R be the mass and the radius of the sphere, respectively.
Calculate the acceleration of the motion of its center.



Solution: When the sphere rolls without slipping, the point of contact moves the distance s = R.α, which is the distance moved by the sphere's center.

The motion can be considered as the rotation of the sphere about the instantaneous axis through the point of contact. Such a motion will be described by the equation

$$\tau = I\frac{d\omega}{dt}$$

where τ = m g x = m g R sinα and I is the moment of inertia of the sphere related to this instantaneous axis that can be calculated according to the parallel-axis theorem

$$I = I_{CM} + mR^2 = \frac{7}{5}mR^2$$

Thus, we get the equation

$$mgR\sin\alpha = \frac{7}{5}mR^2\frac{d\omega}{dt}$$

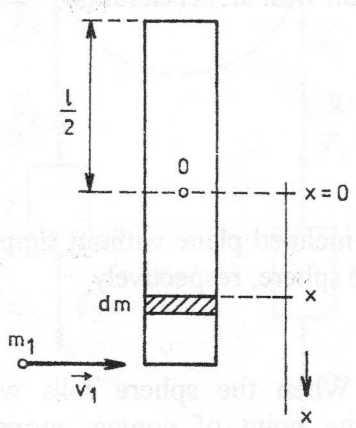
and from here

$$\frac{d\omega}{dt} = \frac{5}{7}\frac{g}{R}\sin\alpha$$

and thus

$$a = R\frac{d\omega}{dt} = \frac{5}{7}g\sin\alpha$$

Problem 1-118. A bar of mass m and length L can rotate about an axis through its center perpendicular to the bar. A projectile of mass m_1 strikes the bar's end with velocity v_1 perpendicular to both the bar and the axis.
Calculate the initial angular velocity of the bar at the moment of striking (the projectile is assumed to be stuck).



Solution: The angular momentum of the projectile with respect to axis 0 equals

$$L_1 = m_1 v_1 \frac{L}{2}$$

The angular momentum of the bar equals

$$L_2 = \int_{(m)} x^2 \omega\, dm = \omega I$$

where I is the moment of inertia of the bar relative to axis 0.

The law of conservation of angular momentum gives us the equation

$$m_1 v_1 \frac{L}{2} = \omega I + m_1 v \frac{L}{2}$$

where v is the initial velocity of the system after striking.

We substitute

$$v = \frac{L}{2}\omega \quad and \quad I = \frac{1}{12}mL^2$$

and we obtain

$$m_1 v_1 \frac{L}{2} = \frac{1}{12}mL^2\omega + m_1\omega\frac{L^2}{4}$$

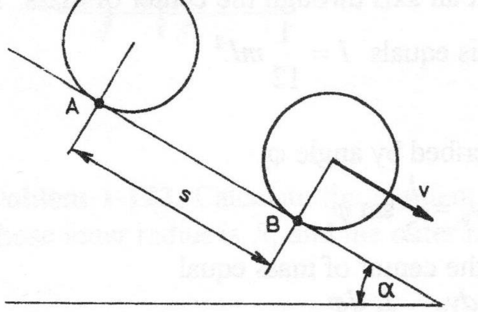
and from here

$$\omega = \frac{6m_1 v_1}{mL + 3m_1 L}$$

Problem 1-119. A uniform cylinder starts to roll without slipping from point A of an inclined plane. Calculate the velocity of the cylinder at point B and the time required to travel the path $s = \overline{AB}$ (see Fig.)

Solution: We write the law of conservation of mechanical energy

$$mgs(\sin\alpha) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$



If we substitute $I = \frac{1}{2}mr^2$ *and* $\omega = \frac{v}{r}$

we get for the velocity at point B

$$v = 2\sqrt{\frac{gs(\sin\alpha)}{3}}$$

The time required to travel path s can be calculated from the definition of velocity

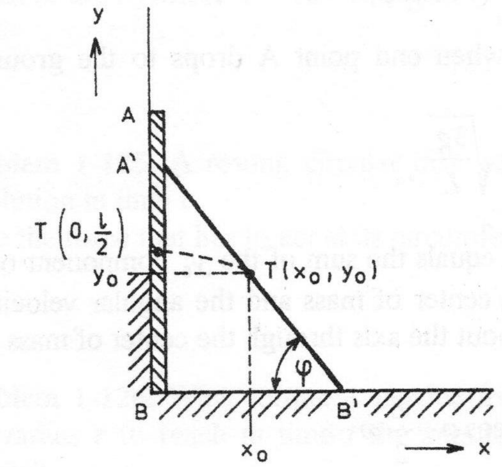
$$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v} = \frac{1}{2}\sqrt{\frac{3}{g\sin\alpha}}\frac{ds}{\sqrt{s}}$$

and after integrating we obtain

$$t = \sqrt{\frac{3s}{g\sin\alpha}}$$

provided $v = 0$ at $t = 0$.

Problem 1-120. A uniform bar of length L bears upon two perfectly smooth walls (see Fig.). The motion of the bar begins from its vertical position. Calculate the velocity of its top end A when it falls on the horizontal plane.



Solution: For $\varphi = \frac{\pi}{2}$ the potential energy of the bar is

$$PE = mg\frac{L}{2} \textit{ and its } KE = 0$$

The motion of the bar is a combination of the translational motion of its center of mass and the rotational motion about the axis through the center of mass.

For any angle $\varphi \neq 0$ the potential energy equals

$$PE = mg\frac{L}{2}\sin\varphi$$

and the kinetic energy equals the sum of the KE of the translational motion and the KE of the rotational motion

$$KE = \frac{1}{2}mv_0^2 + \frac{1}{2}I\omega^2$$

where v_0 is the velocity of the translational motion of the center of mass and ω is the angular velocity of the rotational motion about an axis through the center of mass. The moment of inertia of the bar relative to this axis equals $I = \frac{1}{12}mL^2$.

The location of the center of mass can be described by angle φ:

$$x_0 = \frac{L}{2}\cos\varphi, \quad y_0 = \frac{L}{2}\sin\varphi$$

and the components of the velocity vector of the center of mass equal

$$\frac{dx_0}{dt} = -\frac{L}{2}\frac{d\varphi}{dt}\sin\varphi, \quad \frac{dy_0}{dt} = \frac{L}{2}\frac{d\varphi}{dt}\cos\varphi$$

Then

$$v_0^2 = \frac{L^2}{4}\left(\frac{d\varphi}{dt}\right)^2$$

and the kinetic energy of the bar equals

$$KE = \frac{1}{8}mL^2\left(\frac{d\varphi}{dt}\right)^2 + \frac{1}{24}mL^2\omega^2 = \frac{1}{6}mL^2\omega^2$$

From the law of conservation of energy we can write

$$mg\frac{L}{2} = mg\frac{L}{2}\sin\varphi + \frac{1}{6}mL^2\omega^2$$

and from here

$$\omega = \sqrt{\frac{3g}{L}(1-\sin\varphi)}$$

The angular velocity ω at the moment when end point A drops to the ground is calculated for φ = 0, so

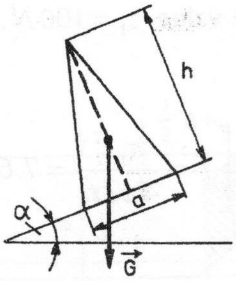
$$\omega = \sqrt{\frac{3g}{L}}$$

The instantaneous velocity of end point A equals the sum of the y_0 component of the velocity of the translational motion of the center of mass and the angular velocity of the rotational motion of the end point A about the axis through the center of mass

$$v_A = \frac{L}{2}\omega\cos\varphi + \frac{L}{2}\omega$$

The velocity of end point A when it drops to the ground equals (φ = 0)

$$v_A = L\omega = \sqrt{3gL}$$

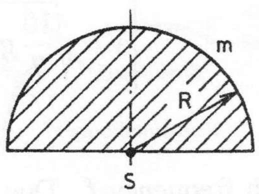


Problem 1-121. There is a squared pyramid on an inclined plane, as shown in Fig. Its height is h and the edge of its base has a magnitude of a.
Calculate the angle for which the pyramid will start to overturn about the edge of its base.

$$\left[\alpha \geq arctg\frac{2a}{h}\right]$$

Problem 1-122. Calculate the moment of inertia of a flywheel whose mass is M and whose inner radius is R_1 and the outer radius equals R_2.

$$\left[I = \frac{1}{2}M\left(R_1^2 + R_2^2\right)\right]$$



Problem 1-123. Calculate the moment of inertia of a uniform half circular plate of mass m and radius R about an axis perpendicular to the plate through its center.

$$\left[I = \frac{1}{2}mR^2\right]$$

Problem 1-124. Calculate the work needed to increase the number of revolutions of a flywheel from the numerical value $n_1 = 333\ \text{min}^{-1}$ *to* $n_2 = 360\ \text{min}^{-1}$, if the moment of inertia of the flywheel $I = 12 \times 10^3\ kg.m^2$.

$$\left[W = 1.3 \times 10^6\ J\right]$$

Problem 1-125. A resting circular disk of mass m and diameter d is to make one revolution in time t.
State the force that has to act at its circumference in a tangential direction.

$$\left[F = \frac{\pi md}{t^2}\right]$$

Problem 1-126. What torque is needed for a resting cylindrical flywheel of mass m and radius r to reach in time t the angular velocity expressed by n revolutions per minute?

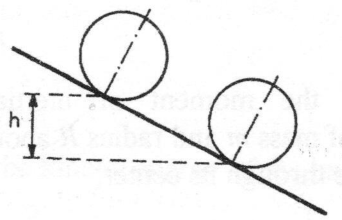
$$\left[\tau = \frac{\pi mr^2 n}{60t}\right]$$

Problem 1-127. A resting flywheel of $I = 540\,kg.m^2$ starts to rotate. The torque acting on it increases linearly with time so that it reaches the value $\tau_1 = 100\,N.m$ at time $t_1 = 10\,s$.
State the frequency of the flywheel at time $t_2 = 72\,s$.

$$\left[f = \frac{\tau_1 t_2^2}{4\pi I t_1^2} = 7.65\,s^{-1} \right]$$

Problem 1-128. Consider a solid disk rolling without slipping down an inclined plane of angle α. Calculate the linear acceleration of its center.

$$\left[a = \frac{2}{3} g \sin\alpha \right]$$



Problem 129. A sphere is rolling without slipping down an inclined plane. Calculate the velocity of its center at the place which is lower of h than the initial position.

$$\left[v = \sqrt{\frac{10}{7} gh} \right]$$

Problem 1-130. A disk of mass m and diameter d rotates with frequency f. Due to braking it will be stopped in time t. Calculate the needed braking torque.

$$\left[\tau = \frac{\pi}{240} \frac{md^2 f}{t} \right]$$

Problem 1-131. A body of mass m_2 is tied to a light string wound around a uniform wheel of mass m_1. The body starts to fall from an unknown height h_x above the ground.
Calculate the height h_x if the body drops to the ground with velocity v.

$$\left[h_x = \frac{v^2 (m_1 + 2m_2)}{4 m_2 g} \right]$$

Problem 1-132. Consider a solid cylinder of mass M and radius R rolling down an inclined plane without slipping. Find the speed of its center of mass when the cylinder reaches the bottom if h is the height of the incline.

$$\left[v = \sqrt{\frac{4}{3} gh} \right]$$