Problem 1-143. Calculate the maximum length of a suspended lead wire for which the wire does not break due to its own weight.
$[\rho = 11.3\times10^3\ kg.m^{-3},\ \sigma_{max} = 19.6\times10^6\ Pa]$

$$[L_{max} = \frac{\sigma_{max}}{g\rho} = 177\,m]$$

Problem 1-144. A steel wire of diameter d has length L if a body of mass m_1 is suspended from it.
Calculate the extension of the wire if we add a body of mass m_2.

$$[\Delta L = \frac{m_2 g\, L}{\frac{\pi d^2}{4}E + m_1 g}]$$

Problem 1-145. A body of mass m is suspended from a steel wire of length L and cross-sectional area S. The body moves in a vertical circle with frequency f.
Determine the instant at which the wire is stretched to its maximum and calculate its maximum extension.

[Answer: the maximum stress comes at the lowest point of the path.]

$$[\Delta L_{max} = \frac{m\,L}{E\,S}\left(g + 4\pi^2 f^2 L\right)]$$

1.8 LIQUID MECHANICS

The density, ρ, of a substance is defined as its mass per unit volume $\rho = \frac{m}{V}$,
where m is the mass of an amount of the substance whose volume is V.
The SI unit for density is $kg.m^{-3}$.

Pressure is defined as the force per unit area, where force F is understood to be acting perpendicular to surface area A :

$$p = \frac{F}{A}$$

The unit of pressure is $N\,m^{-2}$. This unit is the pascal (Pa): 1 Pa = 1 $N\,m^{-2}$.

A liquid exerts a pressure in all directions. The force due to liquid pressure always acts perpendicularly to any surface it is in contact with.
The pressure, or force per unit area, in a liquid at rest is the **hydrostatic force**: at a given point in a liquid at rest, the pressure has the same magnitude regardless of the orientation of the surface on which it acts.

The relation $\frac{dp}{dz} = -\rho g$ (where z is the height above some reference horizontal plane) tells us how pressure varies with height within the liquid. The minus sign indicates that the pressure decreases with an increase in height or that the pressure increases with depth.

If the pressure at height z_1 in the liquid is p_1 and at height z_2 it is p_2, then we can integrate to obtain

$$\int_{p_1}^{p_2} dp = -\int_{z_1}^{z_2} \rho g\, dz \quad or \quad p_2 - p_1 = -\int_{z_1}^{z_2} \rho g\, dz$$

For liquids in which any variation in density can be disregarded, ρ = constant and the equation is readily integrated:

$$p_2 - p_1 = -\rho g(z_2 - z_1)$$

For everyday situations it is convenient to measure distances from the free top surface of a liquid; that is, we let h be the depth in the liquid where $h = (z_2 - z_1)$. If we let z_2 be the pressure of the top surface, then p_2 represents atmospheric pressure p_0 at the top surface. Then for pressure $p(=p_1)$ at depth h in the liquid

$$p = p_0 + \rho g h$$

This shows clearly that the pressure is the same at all points at the same depth.

Pascal's principle: this states that pressure applied to a confined liquid increases the pressure throughout the liquid by the same amount.

Two forces are exerted on an object immersed in a liquid: the force of gravity acting downward and the upward buoyant force exerted by the liquid. The buoyant force arises from the fact that the pressure in a liquid increases with depth. Thus the upward pressure on the bottom surface of a immersed object is greater than the downward pressure on its top surface.

Archimedes' principle: the buoyant force on a body immersed in a liquid is equal to the weight of the liquid displaced by that body.

Archimedes' principle applies equally well to objects that float, such as wood. In general we can say that an object floats on a liquid if its density is less than that of the liquid.

The surface of a liquid is under tension, and this tension, acting parallel to the surface, arises from the attractive forces between the molecules. This effect is called **surface tension**: it is defined as the force F per unit length L that acts across any line in a surface, tending to pull the surface closed:

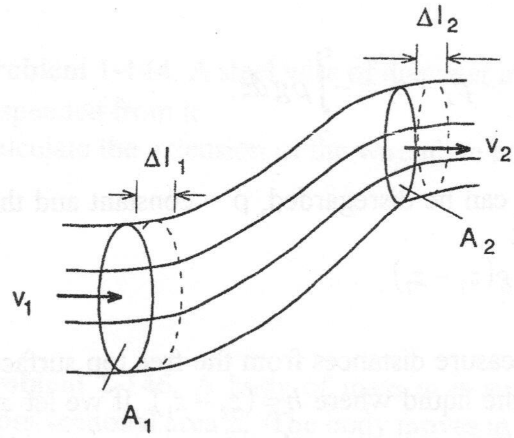
$$\gamma = \frac{F}{L}$$

The SI unit of surface tension is $N\,m^{-1} = kg\,s^{-2}$.

The surface tension is also equal to the work done per unit increase in surface area:

$$\gamma = \frac{W}{\Delta A}$$

Hence, γ can by specified in $N\,m^{-1}$ *or* $J\,m^{-2}$.



In Fig., v_1 represents the liquid's velocity as it passes through the cross-sectional area A_1 and v_2 is the velocity as it passes through the cross-sectional area A_2. The **mass flow rate** is defined as the mass of liquid that passes a given area per unit time:

$$mass\ flow\ rate = \frac{\Delta m}{\Delta t}$$

In Fig., the volume of liquid passing through area A_1 in time Δt is just $A_1\,\Delta L_1$, where ΔL_1 is the distance the fluid moves in time Δt. Since the velocity of the liquid at area A_1 is $v_1 = \frac{\Delta L_1}{\Delta t}$, the mass flow rate through area A_1 is

$$\frac{\Delta m}{\Delta t} = \frac{\rho_1\,\Delta V_1}{\Delta t} = \frac{\rho_1\,A_1\,\Delta L_1}{\Delta t} = \rho_1\,A_1\,v_1$$

where $\Delta V_1 = A_1\,\Delta L_1$ is the volume of mass Δm and ρ_1 is the liquid density at the place of A_1.

Similarly, at area A_2, the flow rate is $\rho_2\,A_2\,v_2$. Since no liquid flows in or out of the sides of a tube of flow, the flow rate through A_1 *and* A_2 must be equal, thus

$$\rho_1\,A_1\,v_1 = \rho_2\,A_2\,v_2$$

This is called the **equation of continuity**.

If the liquid is incompressible, then $\rho_1 = \rho_2$ and the equation of continuity becomes

$$A_1 v_1 = A_2 v_2$$

This equation tells us that where the cross-sectional area is large, the velocity is low and where the area is small, the velocity is high.

Bernoulli's equation:

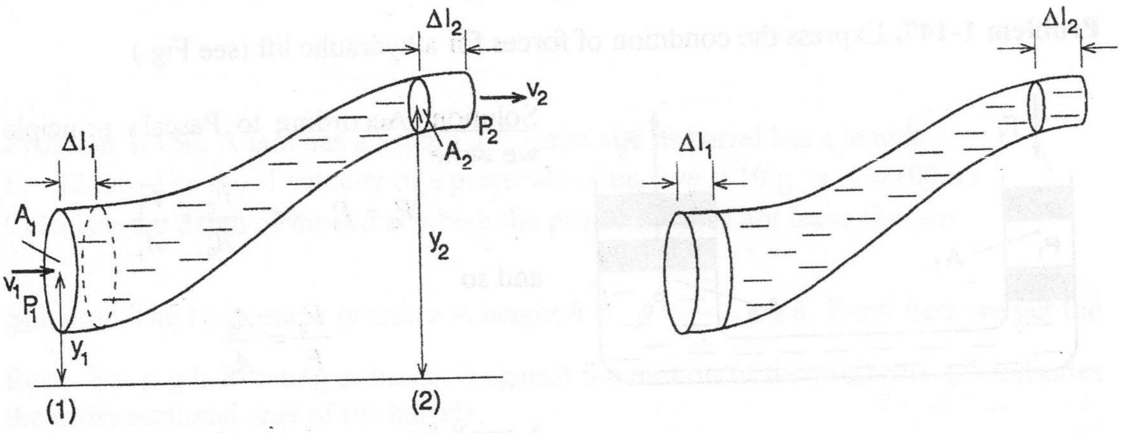
We consider the amount of liquid shown in Fig. and calculate the work done to move it from the position shown in Fig.(a) to that shown in Fig.(b).

We can find work W done on the system by the resultant force as follows:

1. The work done on the system by the pressure force $p_1 A_1$ *is* $p_1 A_1\,\Delta L_1$.
2. The work done on the system by the pressure force $p_2 A_2$ *is* $(-p_2 A_2\,\Delta L_2)$.

Note that it is negative, which means that positive work is done by the system.

3. The work done on the system by gravity is associated with lifting the cross-shaded liquid from height y_1 to height y_2 and is $-mg(y_2 - y_1)$ in which m is the mass of liquid in either cross-shaded area. It too is negative because work is done by the system against the gravitational force.



The work W done on the system by the resultant force is found by adding these three works:

$$W = p_1 A_1 \Delta L_1 - p_2 A_2 \Delta L_2 - mg(y_2 - y_1)$$

According to the work-energy theorem, the net work done on the system is equal to its change in kinetic energy. Thus

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = p_1 A_1 \Delta L_1 - p_2 A_2 \Delta L_2 - mgy_2 + mgy_1$$

The mass m has volume $A_1 \Delta L_1 = A_2 \Delta L_2$. Thus we can substitute $m = \rho A_1 \Delta L_1 = \rho A_2 \Delta L_2$ (we consider the liquid to be incompressible) and after rearranging we obtain

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_1$$

This is **Bernoulli's equation**.
Since points 1 and 2 can be any two points, Bernoulli's equation can be written as

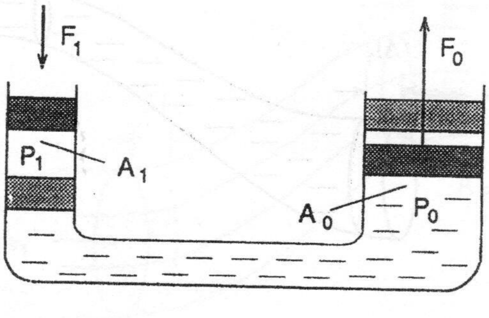
$$p + \frac{1}{2}\rho v^2 + \rho g y = \text{constant}$$

at every point in the liquid.

Problem 1-146. To lift a rock from the bottom of a lake the force F = 400 N was needed. Calculate the mass of the rock if its volume $V = 3 \times 10^4\ cm^3$.

Solution: The buoyant force on the rock due to the water is equal to the weight F_B of water having the volume $V=3\times10^{-2}\,m^3$.
The weight of the rock is $G=mg=F+F_B=F+\rho_{H_2O}Vg=400+300=700$ N and thus the mass of the rock equals $m\approx70$ kg.

Problem 1-147. Express the condition of forces for a hydraulic lift.(see Fig.)



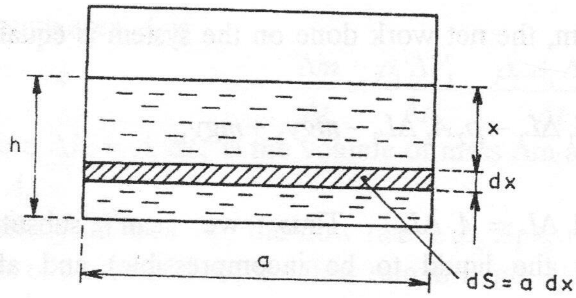
Solution: According to Pascal's principle we write

$$P_0 = P_1 \quad or \quad \frac{F_0}{A_0}=\frac{F_1}{A_1}$$

and so

$$\frac{F_0}{F_1}=\frac{A_0}{A_1}$$

A small force can be used to exert a large force by making the area of one piston (the output) larger than the area of the other (the input).



Problem 1-148. What force does water exert on a vertical wall of an aquarium whose length is a if the water level is at height h?

Solution: The hydrostatic pressure at the chosen surface element dS at depth x equals

$$p_x = \rho g x.$$

The force exerted on the surface element dS is

$$dF = p_x dS = a\rho g x\, dx$$

and the resultant force on the wall is

$$F=\int_{x=0}^{h} dF = \frac{1}{2}a\rho g h^2$$

Problem 1-149. Calculate the increase in pressure inside a pipeline whose length is L = 10 km if the velocity of water $v = 1ms^{-1}$ and if we close the pipeline uniformly during time Δt = 10 s.

Solution: The momentum change of water is $v.\Delta m = \rho$ S L v. The force by which the closure is exerted on the water is

$$F = \frac{v\,\Delta m}{\Delta t} = \frac{\rho S L v}{\Delta t}$$

and thus the increase in pressure equals

$$\Delta p = \frac{F}{S} = \frac{\rho L v}{\Delta t} = 10^6\ Pa$$

Problem 1-150. A gun has a calibre $d = 7$ mm and its barrel has a length $L = 12$ cm. The initial velocity of a projectile of mass $m = 10$ g is $v = 100\,ms^{-1}$. Calculate the depth of the sea at which the projectile does not leave the gun.

Solution: The hydrostatic pressure at height h is $p = \frac{F}{S} = \rho g h$. From here we get the force $F = \rho g h S$ acting at height h against the motion of the projectile. (S indicates the cross-sectional area of the barrel)
The work needed to overcome the force F due to the hydrostatic pressure has to be equal to the kinetic energy of the projectile (the work is done along the path L):

$$\frac{1}{2} m v^2 = F\,L$$

thus we can write

$$\frac{1}{2} m v^2 = \rho g h S L = \frac{1}{4} \rho g h \pi d^2 L$$

and from here

$$h = \frac{2 m v^2}{\rho g \pi d^2 L} = 1071 m$$

Problem 1-151. A 70-kg body of volume $V = 3 \times 10^{-2}\ m^3$ lies on the bottom of a lake. Calculate the force needed to lift it from the lake.

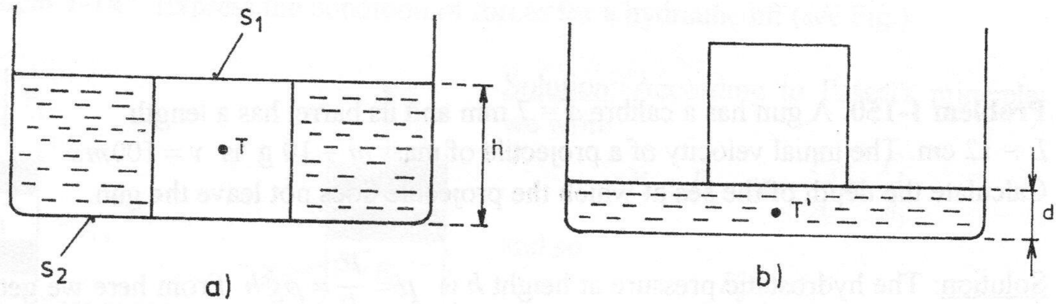
Solution: Force of gravity: $G = m\,g$
Buoyant force $F_B = \rho_{H_2O} V\,g$

The force needed to lift the body equals

$$F = G - F_B = g\left(m - \rho_{H_2O} V\right) = 400 N.$$

Thus, the body seems to have the mass $m' = \frac{F}{g} \approx 40\,kg$.

Problem 1-152. A cylindrical body of base S_1, height h and density ρ is immersed in a cylindrical tank of base S_2 that is filled up to the upper base of the immersed body. (Fig.-a) The density of the liquid is ρ_v.
Calculate the work needed to lift the body above the liquid level. (Fig.-b)



Solution: We shall use the inverse procedure and we immerse the cylinder in a liquid. The original center of mass T' of the liquid will change at position T and the original height of the liquid level d will change to height h.(see Figs.)
Thus, work A' will be done which is equal to the work needed to transfer the mass m_v of liquid from T' to T:

$$A' = m_v g\left(\frac{h}{2}-\frac{d}{2}\right) = (S_2 - S_1)h\rho_v g\left(\frac{h}{2}-\frac{d}{2}\right)$$

For the volume of the liquid we can write

$$(S_2 - S_1)h = S_2 d$$

and thus

$$d = \frac{(S_2 - S_1)}{S_2}h$$

Hence

$$A' = (S_2 - S_1)h\rho_v g\frac{h}{2}\left(1-\frac{S_2 - S_1}{S_2}\right) = (S_2 - S_1)\frac{S_1}{S_2}\frac{h^2}{2}\rho_v g$$

The work needed to lift the body in air equals

$$A'' = mgd = mg\frac{S_2 - S_1}{S_2}h = (S_2 - S_1)\frac{S_1}{S_2}g\rho h^2$$

Thus, the resultant work needed to lift the body above the surface of the liquid is

$$A = A'' - A' = (S_2 - S_1)\frac{S_1}{S_2}gh^2\left(\rho - \frac{\rho_v}{2}\right)$$

Problem 1-153. A dam has a rectangular cross section of height $h = 6$ m. The length of the dam is L = 150 m. The liquid level reaches to the top of the dam.

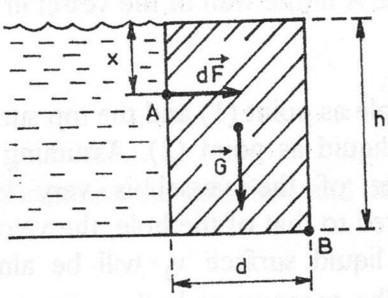
Calculate: 1. The moment of force due to hydrostatic pressure with respect to the bottom edge of the dam.

2. The width of the dam for it to be stable if the density of its material is $\rho = 2400\, kg\, m^{-3}$.

Solution: The hydrostatic pressure at depth x is $p_x = \rho_v\, g\, x$, where ρ_v indicates the density of water.

The force exerted on the surface element Ldx at depth x equals

$$dF = \rho_v\, g\, L\, x\, dx$$



The torque of this force with respect to the point B equals

$$d\tau = (h - x)\, \rho_v\, g\, L\, x\, dx$$

The resultant torque exerted on the dam is now

$$\tau = \int_0^h d\tau = \frac{1}{6}\rho_v\, g\, L h^3 = 5.3 \times 10^7\ N.m$$

The dam will be stable if the moment of the force of gravity τ_G is greater than τ.

Since

$$\tau_G = d\, h\, L \rho g \frac{d}{2}$$

we get the condition

$$\frac{1}{2}\rho g h L d^2 \rangle \frac{1}{6}\rho_v\, g\, L h^3$$

and from here

$$d \rangle h\sqrt{\frac{\rho_v}{3\rho}} = 2.24 m$$

Problem 1-154. Calculate the overpressure inside a soap bubble of radius r caused by surface tension.

Solution: If there is an overpressure Δp inside the soap bubble, then if the bubble is to increase its volume $V = \frac{4}{3}\pi r^3$ of $dV = 4\pi r^2 dr$, work dW is needed to do so:

$$dW = \Delta p\, dV = \Delta p\, 4\pi r^2 dr$$

This work is transformed to the increase in the surface energy dW = γ 2 dS, where γ is the surface tension and dS = 8π r dr is the increase in the bubble's surface $S = 4\pi r^2$ (note that the bubble has two surfaces: inner and outer).

Thus we can write an equality

$$\Delta p 4\pi r^2 dr = \gamma 16\pi r\, dr$$

and from here

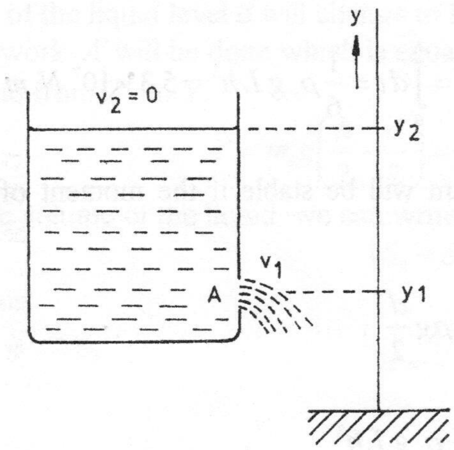
$$\Delta p = \frac{4\gamma}{r}$$

Note: in the case of a drop of liquid we may take the same procedure to solve the problem but we must assume one surface of the drop only and the result will be

$$\Delta p = \frac{2\gamma}{r}$$

Problem 1-155. How fast does water flow from hole A in the wall of the vessel in Fig. filled with water?

Solution: For Bernoulli's equation we choose the hole as point (1) and the top surface of the liquid as point (2). Assuming the diameter of the vessel is very large compared to that of the hole, the velocity of the liquid surface v_2 will be almost zero. The pressure at both points is the same and it is equal to atmospheric pressure, so $p_1 = p_2$.



Thus, for this case Bernoulli's equation may be written as

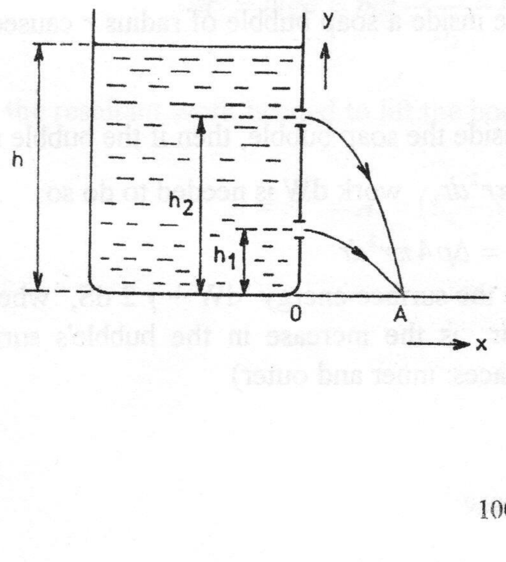
$$\frac{1}{2}\rho v_1^2 + \rho g y_1 = \rho g y_2$$

and thus

$$v_1 = \sqrt{2g(y_2 - y_1)}$$

This is known as Torricelli's formula.

Problem 1-156. There are two openings in the wall of the vessel at heights h_1 *and* h_2 above the bottom.



Calculate the height h of the liquid level so that the liquid from both openings falls into the same place A of the plane of the vessel.

Solution: The path of the liquid after leaving the opening is the path of a horizontal projectile motion. In the chosen coordinate axes this motion is described as follows:

$$x = vt, \quad y = \frac{1}{2}gt^2$$

Therefore, at point A we may write $\qquad v_1 t_1 = v_2 t_2 \qquad (1)$

where v_1, v_2 are the velocities at which the liquid flows out of the openings h_1 *and* h_2, respectively.

From the result of problem 1-155, these velocities are expressed as

$$v_1 = \sqrt{2g(h-h_1)}, \quad v_2 = \sqrt{2g(h-h_2)}$$

The intervals t_1 *and* t_2 are calculated from y-coordinates of the horizontal projection:

$$t_1 = \sqrt{\frac{2h_1}{g}}, \quad t_2 = \sqrt{\frac{2h_2}{g}}$$

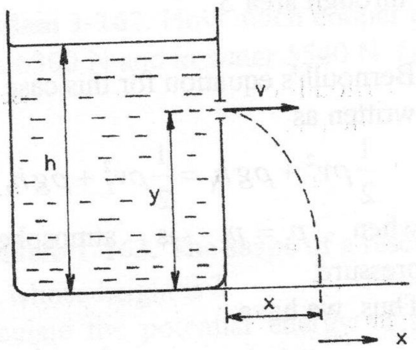
Substituting the velocities and intervals into the condition for point A gives

$$h = h_1 + h_2$$

Problem 1-157. The level of the water in a vessel reaches height *h*. Calculate the position of the opening in the vessel wall so that the water falls on the horizontal plane as long as possible.

Solution: The velocity of water from the opening at height *y* is given (see problem 1-155)

$$v = \sqrt{2g(h-y)}$$



The path of the falling water is practically a horizontal projection whose parametric equations are

$$x = vt, \quad y = \frac{1}{2}gt^2$$

Thus, distance x is given as

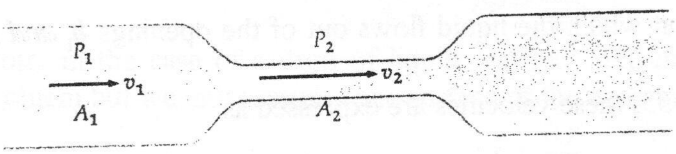
$$x = \sqrt{2g(h-y)}\sqrt{\frac{2y}{g}} = 2\sqrt{y(h-y)}$$

If we solve the maximum of the function $x = f(y)$ we get the answer: $y = \frac{h}{2}$

Problem 1-158. An incompressible liquid such as water flows through a horizontal pipe which has a constricted section as shown in Fig.
Show that the pressure is reduced in the constriction.

Solution: Since both sections of the pipe are at the same elevation, we take $y_1 = y_2$ in Bernoulli's equation. We then have

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2 \qquad (1)$$



But from the equation of continuity with constant density

$$A_1 v_1 = A_2 v_2 \quad or \quad v_2 = \frac{A_1}{A_2} v_1$$

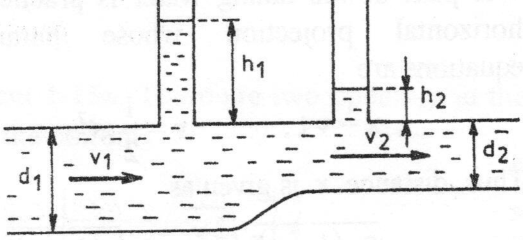
Substituting this result in Eq.(1) gives

$$p_1 - p_2 = \frac{1}{2}\rho\left(\frac{A_1^2}{A_2^2} - 1\right)v_1^2 \qquad (2)$$

Since A_1 is larger than A_2, the quantity on the right is positive and p_2 must be less than p_1. Note that the only circumstance in which p_2 and p_1 can be equal, according to result (2), is when $v_1 = 0$; that is, when the liquid is at rest in the pipe.

Problem 1-159. An incompressible liquid such as water flows through a horizontal pipe that has a constricted section as shown in Fig.
Calculate the amount of water that flows through the pipe in time t if water reaches the levels h_1 *and* h_2 in the manometer tubes.

Solution: The volume of water that flows in time t through any cross-sectional area S_i equals $V = S_i v_i t$, where v_i is the velocity of water through area S_i.



Bernoulli's equation for this case is written as

$$\frac{1}{2}\rho v_1^2 + \rho g h_1 = \frac{1}{2}\rho v_2^2 + \rho g h_2$$

when $p_1 = p_2$ is atmospheric pressure.
Thus, we have

$$v_2^2 - v_1^2 = 2g(h_1 - h_2).$$

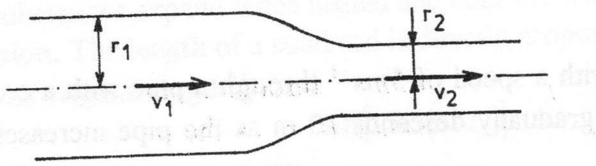
For an incompressible liquid we may write the equation of continuity $v_1 S_1 = v_2 S_2$ and thus we get

$$v_2^2 - v_2^2 \frac{S_2^2}{S_1^2} = 2g(h_1 - h_2)$$

and from here we may calculate v_2.

The amount of water which flows through the pipe in time t will be given as

$$V = S_2 v_2 t = \frac{\pi}{4} t \sqrt{\frac{2g(h_1 - h_2)}{\frac{1}{d_2^4} - \frac{1}{d_1^4}}}$$



Problem 1-160. Calculate the velocity of water through a wider section of a horizontal pipe if the radius of a narrower section is three times less than that of the wider section, and if the pressure difference between the two sections is $\Delta p = 10$ kPa.

[Use the equation of continuity and Bernoulli's equation]

$[v_1 = 0.5 ms^{-1}]$

$[v_2 = 9v_1 = 4.5 ms^{-1}]$

Problem 1-161. A 14.7-kg crown has an apparent weight when immersed in water corresponding to 13.4 kg. Is it made of gold?

[The crown seems to be made of lead]

Problem 1-162. How much cooper and tin does a bronze cube contain if its weight in air is 6300 N and in water 5540 N. ($\rho_{Cu} = 8.8 \times 10^3\ kg\ m^{-3}, \rho_{Sn} = 7.3 \times 10^3\ kg\ m^{-3}$)

[66 % Cu, 34 % Sn]

Problem 1-163. The shape of a reservoir is like a truncated cone, whose radii are $r_1 \rangle r_2$ and whose height is *h*.
Calculate the potential energy of the water present in the reservoir relative to its bottom if the reservoir is full.

$[W_p = \frac{\pi \rho g h^2}{12}(r_1^2 + 3r_2^2 + 2r_1 r_2)]$

Problem 1-164. The top surface of a cylindrical vessel is open. There is a hole of area S at the center of the bottom of the vessel . Through the top of the vessel the water flows into the vessel at the volume rate of flow *V*.
At which height from the bottom is the water level set?

$[h = V^2/(2\ g\ S^2\ ]$

Problem 1-165. Estimate the hydrostatic difference in blood pressure in a person of height 1.83 m between the brain and the foot, assuming that the density of blood is $1.06 \times 10^3\ kg\,m^{-3}$.

$[1.9 \times 10^4\ Pa]$

Problem 1-166. A block of wood floats in water with two-thirds of its volume immersed.
Find the density of the wood. $[6.7 \times 10^2\ kg\,m^{-3}]$

Problem 1-167. Water is moving with a speed of $5 ms^{-1}$ through a pipe with a cross-sectional area of $4 cm^2$. The water gradually descends 10 m as the pipe increases in area to $8 cm^2$.
What is the speed of flow at the lower level?

[v = 2.5 m/s]

Problem 1-168. Find the equivalent force on a wall of height y and width x owing to the pressure of water contained by the wall. Also find the point of application of the equivalent force.

$[F = \frac{1}{2}\rho g x y^2\ at\ height\ \frac{y}{3}]$

Problem 1-169. What is the time in which water will flow out from a full cylindrical vessel whose bottom surface is S and whose height is h. The water flows out through an opening of area S_1 at the bottom of the vessel.

$\left[t = \frac{S}{S_1}\sqrt{\frac{2h}{g}}\right]$