3.2 ELECTROSTATIC FIELD

Two point electric charges exert a force on each other which is given by **Coulomb's law**

$$\mathbf{F}_{1,2} = \frac{1}{4\pi\varepsilon}\frac{Q_1\,Q_2}{r^2}\mathbf{r}^0$$

where $\mathbf{F}_{1,2}$ is the force exerted by the charge Q_1 on the charge Q_2, $\mathbf{r}^0$ is a unit vector of the position vector of the charge Q_2 with respect to the charge Q_1, and ε is the permittivity of medium in which both charges are placed. The permittivity of a medium can be expressed in the following way:

$$\varepsilon = \varepsilon_0\,\varepsilon_r$$

where ε_o is the permittivity of a free space and ε_r is the relative permittivity of the medium.
The electric field vector **E** in any point of the field is given as the ratio of the force **F**, which acts at the given point on a charge Q', to that charge. Near the point electric charge the electric field **E** is given by the expression

$$\mathbf{E} = \frac{\mathbf{F}}{Q'} = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2}\mathbf{r}^0$$

The electric field for a group of charges $Q_1, Q_2 \ldots Q_n$ follows from the principle of superposition

$$\mathbf{E} = \sum \mathbf{E}_k = \frac{1}{4\pi\varepsilon}\sum_{k=1}^{n}\frac{Q_i}{r_k^2}\mathbf{r}_k^0$$

If a **charge distribution is continuous** with a surface charge density σ, the electric field **E** due to this electric charge is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\iint_{(S)}\frac{\sigma\,\mathrm{dS}}{r^2}\mathbf{r}^0$$

where dS is an infinitesimal element of the surface of the conductor and $\mathbf{r}^0$ is a unit vector of a position vector of a point in which we determine the electric field **E** with respect to an elementary surface dS.
If an electric charge is distributed continuously within a certain volume with a volume density ρ the electric field **E** due to this charge is given by the expression

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\iiint_{(V)}\frac{\rho\,\mathrm{dV}}{r^2}\mathbf{r}^0$$

where dV is an infinitesimal element of volume, and $\mathbf{r}^0$ is a unit vector of a position vector of a point in which we determine the electric field with respect to an elementary volume dV.
The flux Φ_E of an electric field through the closed surface is defined as

$$\Phi_E = \iint_{(S)}\mathbf{E}d\mathbf{S}$$

where $d\mathbf{S}$ is a vector normal to the elementary surface dS and oriented outward from the closed surface S.

Gauss' law states that

electric flux through a closed surface is equal to the net charge enclosed by that surface divided by the permittivity of a free space, or

$$\int\int_{(S)} \mathbf{E}d\mathbf{S} = \frac{\sum_{i=1}^{n} Q_i}{\varepsilon_0}$$

The work done by the electric field to move an electric charge Q_0 from point K to point L is equal to

$$W = \int_K^L \mathbf{F}_{el}\,d\mathbf{r} = Q_0 \int_K^L \mathbf{E}d\mathbf{r}$$

If the electric field is produced by a single positive charge Q which is situated in the origin of the reference frame, then the work done to move an electric charge Q_0 from point K to point L is equal to

$$W = \frac{QQ_0}{4\pi\varepsilon_0}\left(\frac{1}{r_K} - \frac{1}{r_L}\right)$$

where r_K and r_L are the initial and final positions of the path travelled by the charge Q_0.
The work done in an electrostatic field to move an electric charge in a round trip is equal to zero

$$\oint \mathbf{E}d\mathbf{r} = 0$$

The potential energy W_p of the point charge Q_0 in the electrostatic field is equal to the work done by the external forces $\mathbf{F}_{ext}$ in moving the charge Q_0 from the reference position B to the point K:

$$W_p = \int_B^P \mathbf{F}_{ext}\,d\mathbf{r} = -\int_B^P \mathbf{F}_{el}\,d\mathbf{r} = -Q_o \int_B^P \mathbf{E}d\mathbf{r}$$

The electric potential (or simply potential) in the point K is defined as the ratio of the potential energy per unit positive charge, or

$$\varphi = \frac{W_p}{Q_0} = -\int_B^P \mathbf{E}dr$$

The potential φ is defined as work done by the external force in moving the unit charge 1 C from the reference position B to the given point P.
In the case of an electrostatic field due to a group of charges the potential in a certain point in this field can be found on the base of the **superposition principle,** or

$$\varphi = \sum_{i=1}^{n} \varphi_i$$

If the charge which is the source of the electrostatic field is continuously distributed over a certain volume or area with a volume charge density ρ or a surface charge density σ respectively, then the potential in the certain point of the field can be expressed as

$$\varphi=\frac{1}{4\pi\varepsilon}\iiint\limits_{(V)}\frac{\rho \mathrm{dV}}{\mathrm{r}}$$

or

$$\varphi=\frac{1}{4\pi\varepsilon}\iint\limits_{(S)}\frac{\sigma \mathrm{dS}}{\mathrm{r}}$$

where r is the distance of the charge dq from the point in which we calculate potential.
The set of all points which have the same potential is called the **equipotential surface.**
The relation between the electric field and the potential in the electrostatic field is expressed in the following way:

$$\mathbf{E} = -\ grad\varphi$$

The electric dipole is a system of two equal but opposite charges $\pm Q$ separated by a distance l.
The electric dipole moment **p**is defined as

$$\mathbf{p} = Q.\mathbf{l}$$

where **l** is a position vector of a positive charge with respect to the negative charge.
An electric dipole placed in a homogenous electric field **E** experiences a torque **D** of the force couple

$$\mathbf{D} = \mathbf{p} \times \mathbf{E}$$

The effect of the torque is to turn the dipole so that the electric dipole moment **p** is parallel to the electric field **E**
In the static situation, when the charges are at rest, the **electric field inside any conductor is equal to zero.** Electric charges are distributed on the surface of the conductor. The surface of the conductor is an equipotential surface. If there is a hole inside a conductor the electric field inside the hole is equal to zero.
The capacitance of the conductor is defined as the ratio of its charge to its potential, or

$$C = \frac{Q}{\varphi}$$

Capacitance depends on the shape and size of the conductor and the material of the surrounding medium.
When a **dielectric** material (insulator) is placed in an external electric field it becomes polarized. Due to the polarization, induced charges appear on the surface of the dielectric material.
The electric polarization vector P is defined as the sum of the induced electric dipole moments per unit volume, or

$$\mathbf{P} = \frac{\sum\limits_{V} \mathbf{p}_i}{V}$$

The surface charge density σ_D of the induced charge is equal to the magnitude of the electric polarization vector, or

$$|\mathbf{P}| = \sigma_D$$

Due to the existence of the induced electric charges the electric field **E** inside the dielectric material is smaller than the external electric field $\mathbf{E}_0$, or

$$\mathbf{E} = \frac{\mathbf{E}_0}{\varepsilon_r}$$

The electric susceptibility is defined as

$$\kappa = \varepsilon_r - 1$$

For a homogeneous and isotropic electric field we can write

$$\mathbf{P} = \varepsilon_0 \kappa \mathbf{E}$$

The electric displacement vector **D** is defined as

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

Gauss's law for a displacement vector states that the flux of the displacement vector depends on the free charge only, or

$$\iint_S \mathbf{D} d\mathbf{S} = Q$$

where Q represents a free charge.

The electrostatic potential energy which is stored in the charged capacitor is

$$W = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C U^2$$

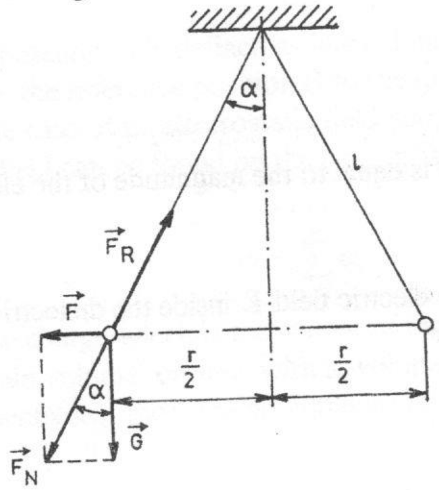
where C is the capacitance of a capacitor and U is the potential difference between the plates of the capacitor.

The electrostatic energy density is defined as the ratio of the energy enclosed in a certain volume, or

$$w_e = \frac{1}{2} \mathbf{E}\,\mathbf{D}$$

Problem 3-14. Two identical, equally charged balls, each of mass 3×10^{-6} kg, are hung from the same hook on strings of length 0.05 m. Their mutual repulsion causes each ball and string to make an angle of 30° with the vertical. What is the charge on each ball?

Solution: As long as the balls are at rest, the net force $\mathbf{F}_N$ acting on each of them acts in the direction of the string. The net force is equal to the vector sum of the electric repulsive force **F** and the weight **G**.



As we can see from the sketch

$$tg\alpha = \frac{F}{G} = \frac{F}{mg}, \text{ and therefore}$$

$$F = mg\ tg\alpha$$

From Coulomb's law

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q\ Q}{r^2}$$

we obtain $$\frac{1}{4\pi\varepsilon_0}\frac{Q\,Q}{r^2} = mg\,tg\alpha$$

For the string we have $\sin\alpha = \frac{r}{2l}$.

Substituting into the equation for balance of forces, we have

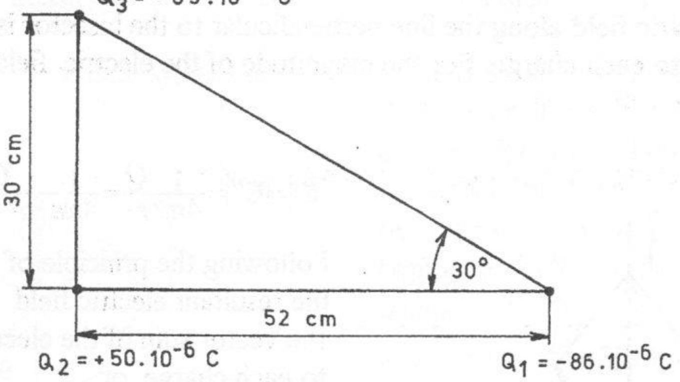
$$\frac{1}{4\pi\varepsilon_0}\frac{Q\,Q}{4\,l^2\sin^2\alpha} = mg\,tg\alpha$$

For the electric charge we obtain

$$Q = 4l\sin\alpha\sqrt{\pi\varepsilon_0\, mg\, tg\alpha} = 4\times 0.05\times\frac{1}{2}\sqrt{\pi\, 8.85\times 10^{-12}\times 3\times 10^{-6}\times 9.81\times\frac{\sqrt{3}}{3}} =$$

$$= 2.2\times 10^{-9}\ C$$

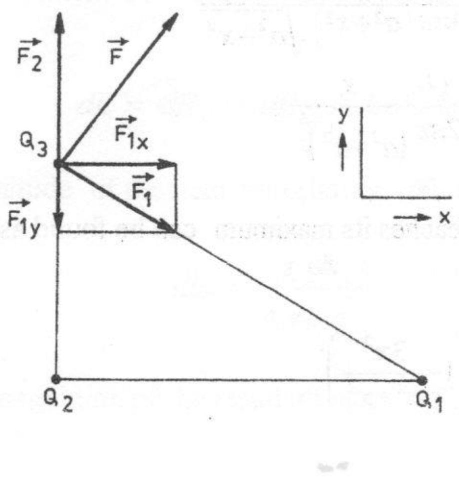
Problem 3-15. Calculate the force on the charge Q_3, which is shown in the Figure, due to the charges Q_1 and Q_2.



Solution: The forces $\mathbf{F}_1$ and $\mathbf{F}_2$ have the directions shown in the diagram, since Q_1 exerts an attractive force and Q_2 a repulsive force. The magnitudes of $\mathbf{F}_1$ and $\mathbf{F}_2$ are (ignoring signs since we know the directions):

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1\,Q_3}{r_{13}^2} = \frac{1}{4\pi\times 8.85\times 10^{-12}}\frac{65\times 10^{-6}\times 86\times 10^{-6}}{(0.3^2+0.52^2)} = 140N$$

$$F_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_2\,Q_3}{r_{23}^2} = \frac{1}{4\pi\times 8.85\times 10^{-12}}\frac{50\times 10^{-6}\times 65\times 10^{-6}}{0.3^2} = 325N$$



We resolve force $\mathbf{F}_1$ into its components along the x and y axes as shown:

$$F_{1x} = F_1\cos 30^0 = 120N$$

$$F_{1y} = -F_1\sin 30^0 = -70N$$

The force $\mathbf{F}_2$ has only a y component. So the net force $\mathbf{F}$ on the charge Q_3 has the components:

$$F_x = F_{1x} = 120\,N$$

$$F_y = F_2 + F_{1y} = 325 - 70 = 255\,N$$

Thus the magnitude of the net force $\mathbf{F}$ is:

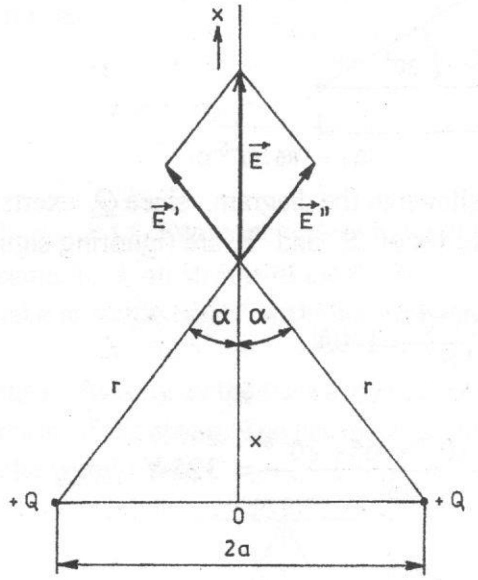
$$F = \sqrt{F_x^2 + F_y^2} = \sqrt{120^2 + 255^2} = 281.8\,N$$

The net force acts at an angle φ given by

$$tg\varphi = \frac{F_y}{F_x} = \frac{255}{120} = 2.12 \quad \text{or} \quad \varphi = 65^0$$

Problem 3-16. Calculate the electric field along the line perpendicular to a bisector of two equal positive charges +Q as a function of the distance from the centre of a bisector. Determine at which distance x the electric field reaches its maximum, and determine the maximum value of the electric field.

Solution: The electric field along the line perpendicular to the bisector is the vector sum of the electric fields due to each charge. For the magnitude of the electric field of each charge we can write:



$$|\mathbf{E}'| = |\mathbf{E}''| = \frac{1}{4\pi\varepsilon}\frac{Q}{r^2} = \frac{1}{4\pi\varepsilon}\frac{Q}{(a^2+x^2)}$$

Following the principle of superposition the resultant electric field $\mathbf{E}$ is equal to the vector sum of the electric fields due to each charge, or

$$\mathbf{E} = \mathbf{E}' + \mathbf{E}''$$

For the magnitude of this electric field we have

$$E = |\mathbf{E}| = 2|\mathbf{E}'|\cos\alpha = 2|\mathbf{E}''|\cos\alpha =$$

$$= 2\frac{1}{4\pi\varepsilon}\frac{Q}{(a^2+x^2)}\frac{x}{\sqrt{a^2+x^2}}$$

$$E = \frac{Q}{2\pi\varepsilon}\frac{x}{(a^2+x^2)^{\frac{3}{2}}}$$

The distance at which the resultant electric field reaches its maximum can be found as an extremum of the function $E=E(x)$, or

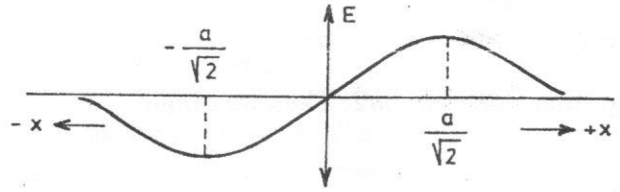
$$\frac{dE}{dx} = \frac{Q}{2\pi\varepsilon}(a^2+x^2)^{\frac{3}{2}}\left(1 - \frac{3x^2}{a^2+x^2}\right)$$

From the condition

$$\frac{dE}{dx} = 0 \qquad \text{we obtain} \quad x_{max} = \pm \frac{a}{\sqrt{2}}$$

The maximum resultant electric field can be found by substituting $x_{max} = \pm \frac{a}{\sqrt{2}}$ into the expression for the electric field. After a little rearrangement we obtain

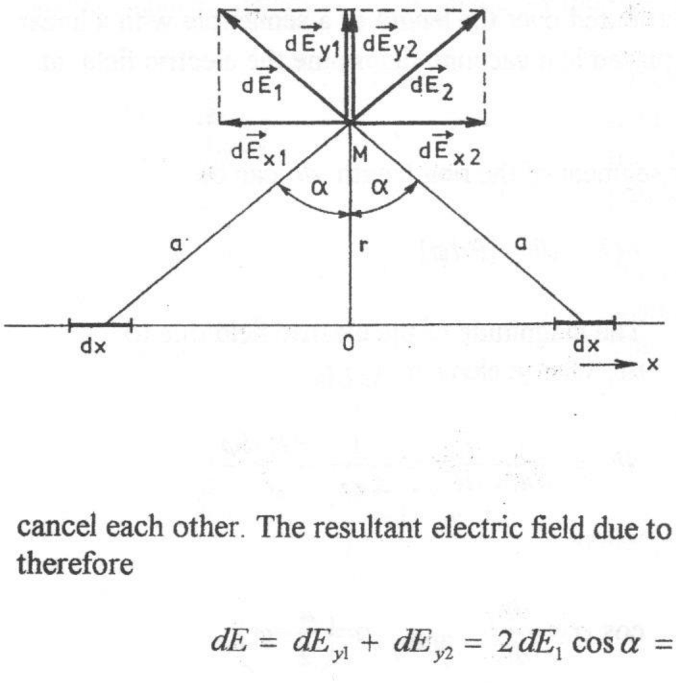
$$E_{max} = \pm \frac{Q}{3\sqrt{3}\pi\varepsilon a^2}$$



The electric field along the line perpendicular to a bisector of two equal charges as a function of the distance from the centre of a bisector is shown in this figure.

Problem 3-17. A charge is distributed uniformly over the entire length of an infinitely long wire with linear charge density τ. Find the electric field **E** at a distance r from the wire.

<u>Solution</u>: We set up a co-ordinate system so the wire is on the x axis with origin O as shown.



A segment of wire dx has the charge $dQ = \tau . dx$ The electric field $d\mathbf{E}_1$ at point M can be resolved into a horizontal component $d\mathbf{E}_{x1}$ and a vertical component $d\mathbf{E}_{y1}$. Symmetrically to the origin there is on the segment dx of the wire an equal electric charge which produces at point M an electric field $d\mathbf{E}_2$, which can also be resolved into two components $d\mathbf{E}_{x2}$ and $d\mathbf{E}_{y2}$. The components $d\mathbf{E}_{x1}$ and $d\mathbf{E}_{x2}$ have the same magnitude but opposite directions Therefore they will cancel each other. The resultant electric field due to the symmetrically placed charges dQ is therefore

$$dE = dE_{y1} + dE_{y2} = 2\,dE_1 \cos\alpha = 2\,dE_2 \cos\alpha$$

The magnitude of the field contribution $d\mathbf{E}_1$ due to the charge element dQ is given as

$$dE_1 = \frac{\tau\, dx}{4\pi\varepsilon\, a^2}$$

and the magnitude of the resultant electric field dE can be expressed as

$$dE = 2\frac{\tau}{4\pi\varepsilon}\frac{\cos\alpha}{a^2}dx$$

The electric field due to the whole charged wire is found by the integration of the previous equation, or

$$E=\frac{\tau}{2\pi\varepsilon}\int_0^{\infty}\frac{\cos\alpha}{a^2}dx$$

This integral is readily evaluated when we use α as a variable. Using the substitution

$$x = r\, tg\,\alpha \qquad dx = \frac{r}{\cos^2\alpha}d\alpha \qquad \text{and} \qquad a = \frac{r}{\cos\alpha}$$

Note that we have to change the limits of integration from 0 to π/2. Thus we obtain

$$E=\frac{\tau}{2\pi\varepsilon}\int_0^{\infty}\frac{\cos\alpha}{a^2}dx=\frac{\tau}{2\pi\varepsilon r}\int_0^{\frac{\pi}{2}}\cos\alpha\, d\alpha=\frac{\tau}{2\pi\varepsilon r}$$

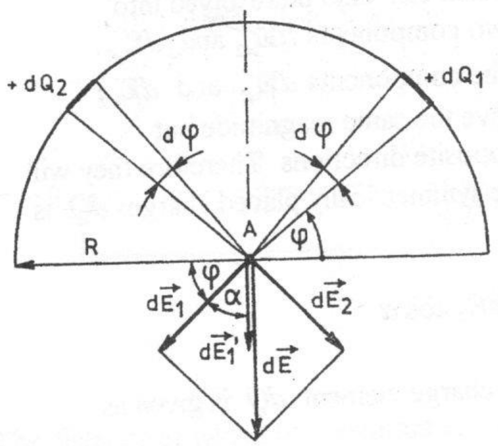
From this result we see that the electric field of a long straight charged wire decreases inversely as the first power of the distance from the wire. This result, obtained for an infinite wire, is a good approximation for a wire of finite length as long as r is small compared to the distance of point M from the ends of the wire.

Problem 3-18. A thin rod, carrying a total charge $+Q$, is bent into the shape of a semicircle with a radius R. The charge is uniformly distributed over the length of a semicircle with a linear charge density τ. Supposing that the rod is placed in a vacuum, determine the electric field at the point A.

Solution: The charge dQ_1 of the particular segment of the ring length dl can be expressed as

$$dQ_1=\tau dl=\tau(R d\varphi)$$

The magnitude of the electric field due to this charge element dQ_1 is



$$dE_1 = \frac{1}{4\pi\varepsilon_0}\frac{dQ_1}{R^2} = \frac{1}{4\pi\varepsilon_0}\frac{\tau R\, d\varphi}{R^2}$$

where

$$\cos\alpha = \frac{dE_1'}{dE_1} \quad \text{and} \quad \alpha=\left(\frac{\pi}{2}-\varphi\right)$$

$$\cos\alpha=\cos\left(\frac{\pi}{2}-\varphi\right)=\sin\varphi$$

$$dE_1' = dE_1\cos\alpha = dE_1\sin\varphi = \frac{1}{4\pi\varepsilon_0}\frac{\tau R\, d\varphi}{R^2}\sin\varphi$$

Since symmetrically to the charge dQ_1 is on the ring placed the same charge dQ_2 which produces at point A the electric field dE_2 than the electric field due to these two charges is

$$dE = 2\,dE_1' = \frac{2\tau R \sin\varphi}{4\pi\varepsilon_0 R^2}\,d\varphi$$

The total charge of the ring is $dQ = \tau\pi R$.

$$dE = \frac{2\tau R \sin\varphi}{4\pi\varepsilon_0 R^2}\,d\varphi\,\frac{\pi}{\pi} = \frac{2Q\sin\varphi}{4\pi^2\varepsilon_0 R^2}\,d\varphi$$

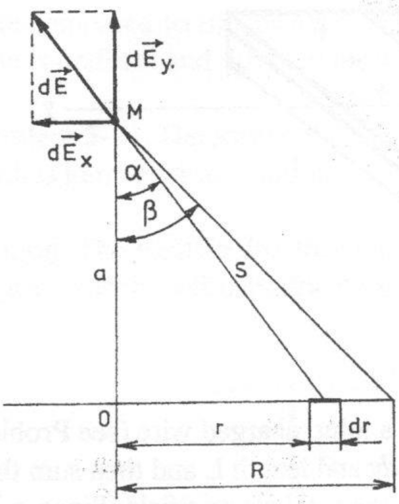
The resultant electric field at point A is found by integrating the effects of all the elements that make up the semicircle, or

$$E = \frac{2Q}{4\pi^2\varepsilon_0 R^2}\int_0^{\frac{\pi}{2}} \sin\varphi\, d\varphi = \frac{2Q}{4\pi^2\varepsilon_0 R^2}$$

Problem 3-19. . Determine the electric field at point M on the axis of a uniformly charged disc of radius R. The surface charge density is σ.

Solution: We divide the disc into thin rings which has the radii r and $r+dr$. As long as the surface charge density is σ the charge of the ring is

$$dQ = 2\pi\, r\sigma dr\,.$$



This electric charge sets up an electric field at point M. Due to the symmetry, the components of the electric field parallel to the plane of the disc are cancelled out by an equal but opposite component established by the charge element on the opposite side of the ring. Thus the component of the electric field perpendicular to the plane of the circle contributes to the resultant electric field, or

$$dE_y = dE\cos\alpha$$

Substituting for

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dQ}{s^2}$$

and integrating from 0 to R we obtain

$$E_y = \frac{1}{4\pi\varepsilon_0}\int_0^R \frac{\cos\alpha}{s^2}\,dQ$$

This integral is readily evaluated when we use α as a variable.

Using the substitutions

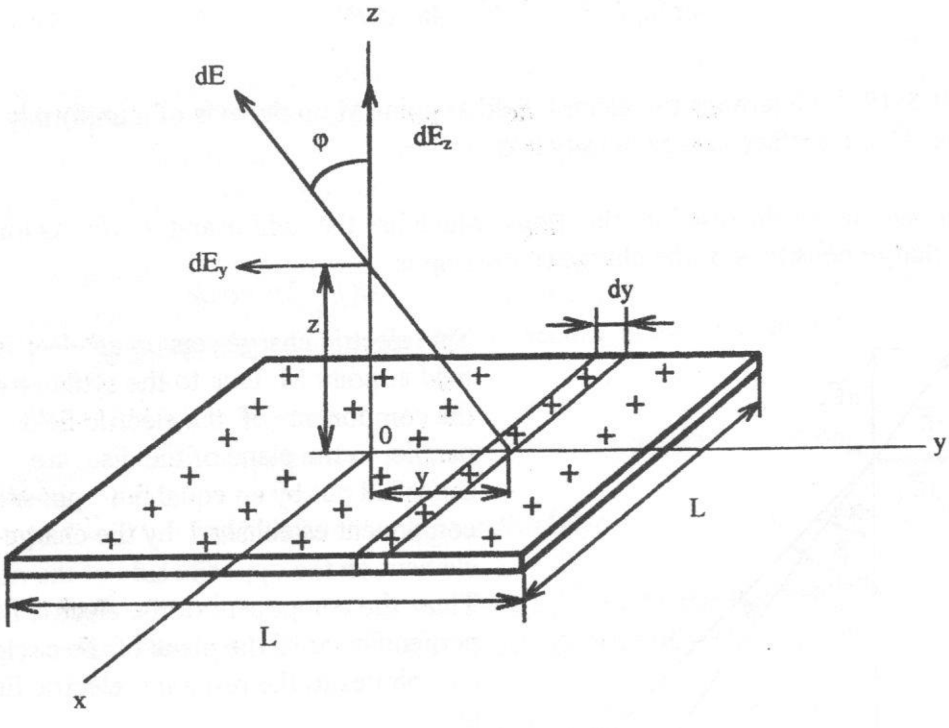
$$s=\frac{a}{\cos\alpha} \qquad r=a\,tg\alpha \qquad dr=\frac{a}{\cos^2\alpha}d\alpha$$

we obtain for the resultant electric field

$$E=E_y=\frac{1}{4\pi\varepsilon}\int\frac{\cos\alpha}{s^2}dQ=\frac{\sigma}{2\varepsilon}\int_0^\beta \sin\alpha d\alpha=\frac{\sigma}{2\varepsilon}\left[-\cos\alpha\right]_0^\beta=\frac{\sigma}{2\varepsilon}\left[1-\cos\beta\right]=$$

$$=\frac{\sigma}{2\varepsilon}\left(1-\frac{a}{\sqrt{a^2+R^2}}\right)$$

Problem 3-20. Electric charge is distributed uniformly over a large square plane of side L, as shown in the following figure. The surface charge density is σ. Calculate the electric field at point P, a distance z above the centre of the plane where z is much less than L.



Solution: Since we already know the electric field due to a long charged wire (see Problem 3-17), let us divide the plane into long narrow strips of width dy and length L and then sum the contributions to the electric field due to each strip to get the total electric field. Since σ is the surface charge density the charge on each strip is

$$dQ=\sigma L\,dy$$

Therefore each strip can be considered as a line of charge with a charge per unit length of

$$\lambda=\frac{dQ}{L}=\frac{\sigma L\,dy}{L}=\sigma dy$$

The distance from point P to the centre of the strip is $\left(x^2+y^2\right)^{\frac{1}{2}}$.

Using the result of Problem 3-17 the electric field due to the strip is

$$dE=\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{\left(x^2+y^2\right)^{\frac{1}{2}}}=\frac{1}{2\pi\varepsilon_0}\frac{\sigma dy}{\left(x^2+y^2\right)^{\frac{1}{2}}}$$

The plane is symmetric about a line through the centre, so when we sum over all the strips that make up the plane, the y components of the electric field will vanish. Hence we need sum over only y components of the electric field, where

$$dE_z=dE\cos\theta=dE\frac{z}{\left(z^2+y^2\right)^{\frac{1}{2}}}$$

The total electric field is thus

$$E=\int dE_z=\frac{\sigma z}{2\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{dy}{\left(z^2+y^2\right)}=\frac{\sigma z}{2\pi\varepsilon_0}\left(\frac{1}{z}arctg\frac{y}{z}\right)_{-\frac{L}{2}}^{\frac{L}{2}}$$

If we consider only points for which $z \ll L$, distant contributions will be small and we can effectively let

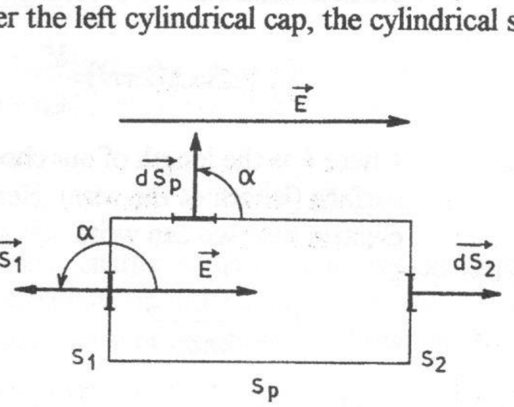
$y=\pm\frac{L}{2}\to\infty$ compared to z, so for the resultant electric field we obtain

$$E=\frac{\sigma}{2\pi\varepsilon_0}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{\sigma}{2\varepsilon_0}$$

This result is valid for any point near an infinite plane. It is also valid for points close to a finite plane compared to the distance to the plane's edge. Thus the field near a large, uniformly charged plane is uniform and directed outward for positive charge and inward for negative charge.

Problem 3-21. Determine the electric flux through a hypothetical closed cylinder of radius R which is immersed in a uniform electric field , the cylinder axis being parallel to the field lines.

Solution: The electric flux through the closed cylinder can be written as the sum of three terms, an integral over the left cylindrical cap, the cylindrical surface and the right cap:



$$\Phi_E=\iint_S \mathbf{E}d\mathbf{S}=$$

$$=\iint_{S_2}\mathbf{E}d\mathbf{S}_2+\iint_{S_p}\mathbf{E}d\mathbf{S}_p+\iint_{S_1}\mathbf{E}d\mathbf{S}_1$$

For the left cap, the angle α for all points is π, the electric field has a constant value, and thus for the flux through the left cylinder cap we can write

$$\iint_{S_1} \mathbf{E}d\mathbf{S}_1 = -ES_1$$

Similarly for the flux through the right cylinder cap we have

$$\iint_{S_2} \mathbf{E}d\mathbf{S}_2 = ES_2$$

Finally for the cylinder wall we have

$$\iint_{S_P} \mathbf{E}d\mathbf{S}_P = 0$$

Problem 3-22. A very long straight wire possesses a uniformly distributed electric charge with

Thus the total flux through the whole cylinder is

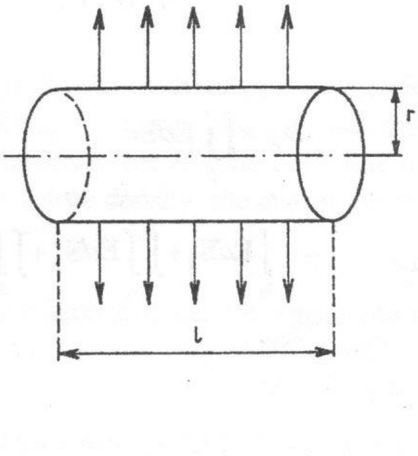
$$\iint_{S} \mathbf{E}d\mathbf{S} = -ES_1 + ES_2 = 0$$

because the areas of the caps are the same. We expected this result because there are no sources or sinks of electric field, that is charges, within the closed surface.

linear charge density ξ. Determine the electric field at points near the wire using Gauss' law.

Solution: Because of the symmetry, we expect the electric field to be directed radially outward provided that the charge is positive, and to depend only on the perpendicular distance r from the wire. Because of this cylindrical symmetry the field will be the same at all points on a Gaussian surface that is a cylinder with a wire along its axis. The electric field is perpendicular to this surface at all points. However for Gauss' law we need a closed surface so we must include the flat ends of the cylinder. Since the electric field is parallel to its ends

$$\iint_{S} \mathbf{E}d\mathbf{S} = E(2\pi r l) = \frac{l\xi}{\varepsilon_0}$$



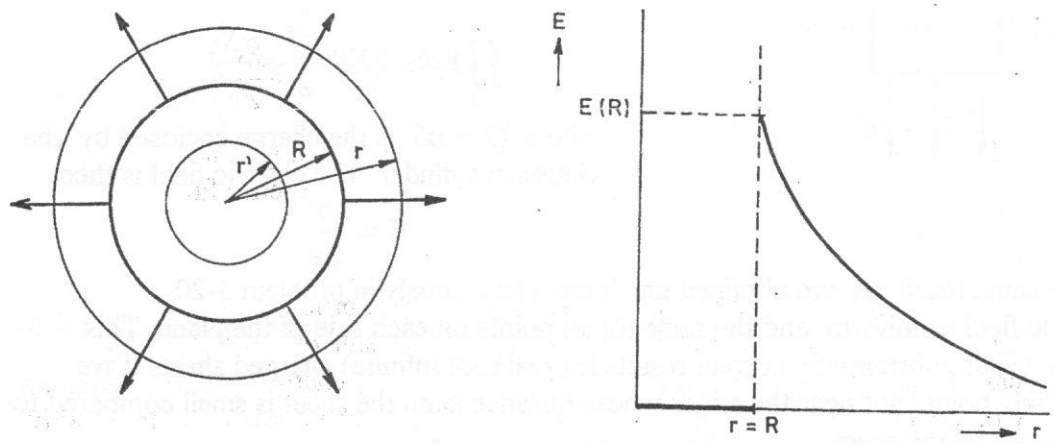
where l is the length of our chosen Gaussian surface (length of the wire). Hence for the electric field we can write

$$E = \frac{1}{2\pi\varepsilon_0}\frac{\xi}{r}$$

This is the same result as we obtained in problem 3-17 using Coulomb's law.

Problem 3-23. An electric charge Q is distributed uniformly on a conducting sphere of radius R. Determine the electric field outside and inside the sphere.

Solution: Since the electric charge is distributed symmetrically, the electric field at all points must also be symmetric. The electric field is directed radially outward for the positive charge.



To determine the electric field outside the sphere, we choose as Gaussian surface a sphere of radius r ($r>R$). As long as the electric field on this sphere is constant we obtain from Gauss' law:

$$\iint_S \mathbf{E}d\mathbf{S}=\iint_S EdS=E\iint_S dS=E4\pi r^2=\frac{Q}{\varepsilon}$$

or

$$E=\frac{1}{4\pi\varepsilon}\frac{Q}{r^2}$$

We see that the field outside a charged sphere is the same as that for a point charge of the same magnitude located at the centre of the sphere.

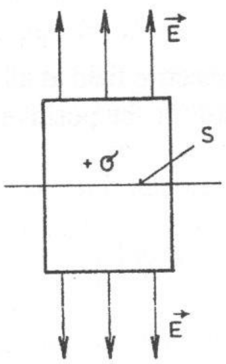
To determine the electric field inside the sphere, we choose for our Gaussian surface a concentric sphere of radius r' ($r'<R$). Thus we can write

$$E\,4\pi r'^2=0$$

because the electric charge is distributed only on the outer surface of the sphere. Thus the electric field inside the conducting sphere is equal to zero. The electric field as a function of distance from the centre of the sphere is shown in the figure.

Problem 3-24. Determine the electric field near an infinite charged conducting plate. The positive charge is distributed uniformly with a surface charge density σ.

Solution: We choose as our Gaussian surface a small closed cylinder whose axis is perpendicular to the plane and which extends through the plane as shown in the figure.



Because of the symmetry, the electric field points at right angles to the end caps and away from the plane. Since **E** does not pierce the cylindrical surface, there is no contribution to the flux from this source and all the flux is through the two end caps. So Gauss' law gives:

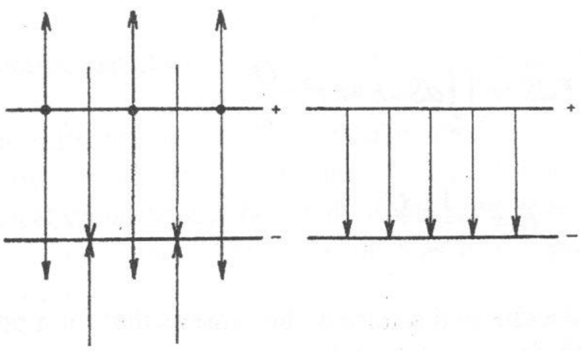
$$\iint_S \mathbf{E}d\mathbf{S} = 2ES = \frac{Q}{\varepsilon_0} = \frac{\sigma S}{\varepsilon_0}$$

where $Q = \sigma S$ is the charge enclosed by the Gaussian cylinder. The electric field is then

$$E = \frac{\sigma}{2\varepsilon}$$

This is the same result as we obtained much more laboriously in problem 3-20. The electric field is uniform and the same for all points on each side of the plane. This derivation yields substantially correct results for real (not infinite) charged sheets if we consider only points not near the edges whose distance from the sheet is small compared to the dimensions of the sheet.

Problem 3-25. Determine the electric field between and outside two very large conducting plates carrying an equal but opposite charge which is distributed over the plates with a surface charge density σ.



Solution: We can solve this problem using the superposition principle. In the region between the plates, the fields of the two plates are in the same directions. Adding the parallel vectors, we obtain

$$E = E_+ + E_- = 2\frac{\sigma}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

In the regions outside the pair of plates, the fields due to each plate are in opposite directions, so the electric field vanishes outside the region between the plates.

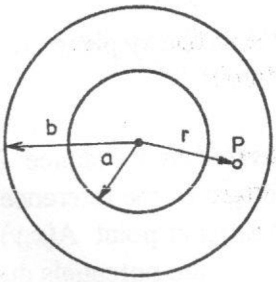
Problem 3-26. Determine the work that must be done to move a positive charge $Q = 3.10^{-8}$ C from a point A with potential $\varphi_A = 300$ V to a point B with potential $\varphi_B = 1200$ V.

Solution: The work that must be done by the external force will be equal to the increase of potential energy of the transported charge. Thus we have

$$A = W_B - W_A = Q\varphi_B - Q\varphi_A = 3\times10^{-8}(1200-300) = 27\times10^{-6}J$$

Problem 3-27. Determine the potential at point P between two coaxial conducting cylinders.

Solution: The electric field between two cylinders can be found from Gauss' law



$$\iint_{(s)} \mathbf{E}d\mathbf{S} = \frac{\sum Q}{\varepsilon}$$

For our case we obtain

$$E\,2\pi r\,l = \frac{Q}{\varepsilon}$$

or

$$E = \frac{Q}{2\pi\,\varepsilon\,r\,l}$$

As long as we can write $E = -\dfrac{d\varphi}{dr}$

we have

$$-\frac{d\varphi}{dr} = \frac{1}{2\pi\varepsilon}\frac{Q}{l\,r} \qquad \text{or} \qquad d\varphi = -\frac{Q}{2\pi\varepsilon}\frac{dr}{l\,r}$$

After integration we have

$$\varphi_r = -\frac{Q}{2\pi\,\varepsilon\,l}\ln r + K$$

Substituting for $r = a$ we obtain

$$\varphi_a = -\frac{Q}{2\pi\,\varepsilon\,l}\ln a + K$$

and for $r = b$ we have

$$\varphi_b = -\frac{Q}{2\pi\,\varepsilon\,l}\ln b + K$$

Eliminating K and $\dfrac{Q}{2\pi\,\varepsilon\,l}$ we have

$$\varphi_b - \varphi_a = -\frac{Q}{2\pi\,\varepsilon\,l}\ln\frac{b}{a}$$

$$K = \varphi_a + \frac{Q}{2\pi\,\varepsilon\,l}\ln a$$

Thus we obtain for potential at point P

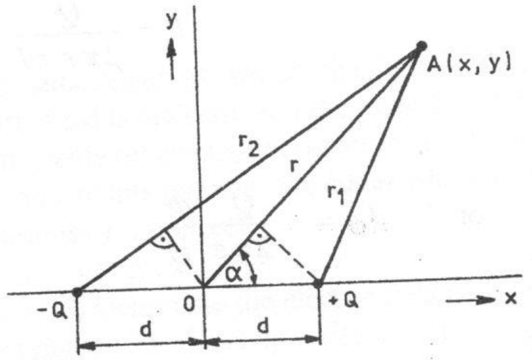
$$\varphi_r = \frac{(\varphi_b - \varphi_a)}{\ln\dfrac{b}{a}}\ln r + \varphi_a - \frac{(\varphi_b - \varphi_a)}{\ln\dfrac{b}{a}}\ln a$$

after a little rearrangement we finally have

$$\varphi_r = \varphi_a + (\varphi_b - \varphi_a)\frac{\ln\frac{r}{a}}{\ln\frac{b}{a}}$$

Problem 3-28. Determine the electric field at any point A in the *xy* plane due to an electric dipole. Assume that the point A is not too close to the dipole.

Solution: Two equal charges of opposite sign, $\pm Q$, separated by a distance $2d$, constitute an electric dipole (see the following figure). We place the origin of the reference frame into the middle of the line joining the two charges. Since the potential at point $A(x,y)$ is the sum of the potentials due to each of the two charges, we have



$$\varphi = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

We can express the distances r_1 and r_2 from point A to the positive and negative charge respectively, as a function of x and y as

$$r_1 = \sqrt{(x-d)^2 + y^2} \qquad r_2 = \sqrt{(x+d)^2 + y^2}$$

Substituting for r_1 and r_2 into the expression for potential we have

$$\varphi = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{(x-d)^2 + y^2}} - \frac{1}{\sqrt{(x+d)^2 + y^2}}\right)$$

We find E_x and E_y recalling that

$$E_x = -\frac{d\varphi}{dx} \qquad E_y = -\frac{d\varphi}{dy}$$

After derivation we therefore obtain

$$E_x = \frac{Q}{4\pi\varepsilon}\left(-\frac{x+d}{\left[(x+d)^2 + y^2\right]^{\frac{3}{2}}} + \frac{x-d}{\left[(x-d)^2 + y^2\right]^{\frac{3}{2}}}\right)$$

$$E_y = \frac{Q}{4\pi\varepsilon}\left(-\frac{y}{\left[(x+d)^2 + y^2\right]^{\frac{3}{2}}} + \frac{y}{\left[(x-d)^2 + y^2\right]^{\frac{3}{2}}}\right)$$

Of practical importance are the points whose distance from the dipole is much larger than the separation of those two charges, that is, for $r \gg l$. In this case we obtain for the distances r_1 and r_2:

$$r_1 \approx r - d\cos\alpha \qquad\qquad r_2 \approx r + d\cos\alpha .$$

Substituting these equations into the expression for potential we have

$$\varphi = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{r - d\cos\alpha} - \frac{1}{r + d\cos\alpha}\right)$$

after a little rearrangement we obtain

$$\varphi = \frac{Q}{4\pi\varepsilon r}\left(\frac{1}{1-\dfrac{d\cos\alpha}{r}} - \frac{1}{1+\dfrac{d\cos\alpha}{r}}\right)$$

Taking into account that $r \rangle\rangle d$ then $\dfrac{d\cos\alpha}{r} \langle\langle 1$. Taking the last inequality into account, we can write the binomial theorem in the form

$$(1 \pm x)^{-1} = 1 \mp nx$$

Thus the expression for potential reduces to $\varphi = \dfrac{Q}{4\pi\varepsilon}\dfrac{2d\cos\alpha}{r^2}$

To determine the expression for the electric field in the x,y co-ordinates we put $2\,s\,Q = p$ and

$$\cos\alpha = \frac{x}{\sqrt{x^2 + y^2}} .$$

For potential we therefore have

$$\varphi = \frac{px}{4\pi\varepsilon\sqrt{(x^2 + y^2)^3}}$$

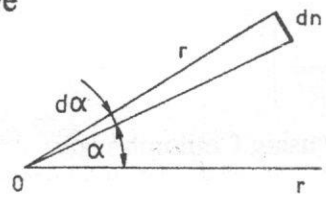
For the components of the electric field we obtain

$$E_x = -\frac{d\varphi}{dx} = \frac{p\left(2x^2 - y^2\right)}{4\pi\varepsilon\left(x^2 + y^2\right)^{\frac{5}{2}}} \qquad\qquad E_y = -\frac{d\varphi}{dy} = \frac{3pxy}{4\pi\varepsilon\left(x^2 + y^2\right)^{\frac{5}{2}}}$$

For the magnitude of the electric field we have

$$E = \sqrt{E_x^2 + E_y^2} = \frac{p}{4\pi\varepsilon}\frac{\sqrt{\left(2x^2 - y^2\right)^2 + 9x^2y^2}}{\left(x^2 + y^2\right)^{\frac{5}{2}}}$$

We can also express the radial and tangential component of the electric field (see following figure), from the potential function $\varphi(r,\alpha)$. Taking its partial derivatives we have



$$E_r = -\frac{d\varphi}{dr} = \frac{2p\cos\alpha}{4\pi\varepsilon r^3}$$

for the radial component of the electric field.

Taking into account that $dn = r\,d\alpha$ we can determine the tangential component of the electric field as

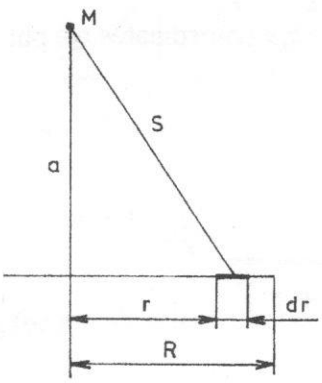
$$E_n = -\frac{d\varphi}{dn} = -\frac{1}{r}\frac{d\varphi}{d\alpha} = \frac{p\sin\alpha}{4\pi\varepsilon\, r^3}$$

Finally for the resultant electric field we find

$$E = \sqrt{E_r^2 + E_n^2} = \frac{p}{4\pi\varepsilon\, r^3}\sqrt{\sin^2\alpha + 4\cos^2\alpha}$$

Problem 3-29. A thin flat disc of radius R carries a uniformly distributed charge Q. Determine the potential at point M on the axis of the disc, a distance a from its centre. The surface charge density is σ.

Solution: We divide the disc into thin rings of radius r and thickness dr. The charge Q is distributed uniformly, so the charge contained in each ring is



$$dQ = 2\pi r\sigma\, dr.$$

All parts of this charge element are at the same distance s from point M so that their contribution $d\varphi$ to the electric potential at M is given by the expression

$$d\varphi = \frac{1}{4\pi\varepsilon}\frac{2\pi r\,\sigma}{s}$$

The potential φ at point M is found by integrating over all the strips into which the disk can be divided, or

$$\varphi = \frac{\sigma}{2\varepsilon}\int_0^R \frac{r}{s}\,dr$$

This integral is readily evaluated when we use s as a variable. As long as we can write

$$s^2 = a^2 + r^2$$

we obtain for the resultant potential

$$\varphi = \frac{\sigma}{2\varepsilon}\int_0^R \frac{r}{\sqrt{a^2 + r^2}}\,dr = \frac{\sigma}{2\varepsilon}\int_a^{\sqrt{a^2+R^2}} \frac{s\,ds}{s} = \frac{\sigma}{2\varepsilon}\left(\sqrt{a^2 + R^2} - a\right)$$

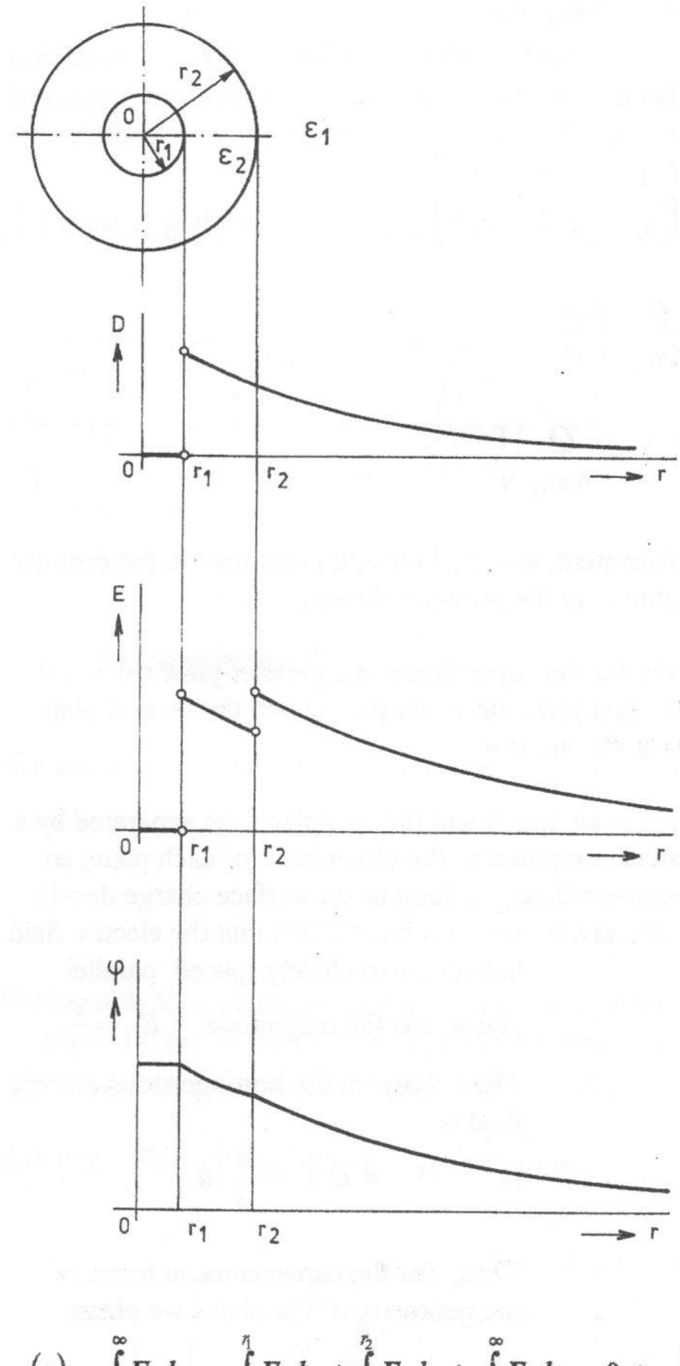
Because of the symmetry it is obvious that the electric field will have the same direction as the axis of the symmetry (oriented outward for a positively charged ring). For its magnitude we obtain

$$E = -\frac{d\varphi}{da} = \frac{\sigma}{2\varepsilon}\left(1 - \frac{a}{\sqrt{a^2 + R^2}}\right)$$

This is the same result as we obtained in problem 3-19 using Coulomb's law.

Problem 3-30. A particular conducting sphere of radius r_1 is surrounded by a spherical dielectric with the inner radius r_1 and the outer radius r_2. The permittivity of the dielectric is ε_2 and the permittivity of the surrounding medium is ε_1 where $\varepsilon_1 \langle \varepsilon_2$. Find the dependence of electric displacement *D*, electric field *E* and potential φ (with respect to infinity) on the distance from the centre of the sphere. The charge of the conducting sphere is *Q*.

Solution:



a). Electric displacement:

From Gauss' law we have for $r<r_1$:

$$D(r) = 0$$

for $r>r_1$:

$$\iint_{(S)} \mathbf{D}\, d\mathbf{S} = D\iint dS_n = $$

$$= D\, 4\pi r^2 = Q$$

or

$$D(r) = \frac{Q}{4\pi r^2}$$

b). Electric field:

for $r<r_1$: $\quad E(r) = 0$

for $r_1 \langle r \langle r_2$

$$E(r) = \frac{Q}{4\pi\varepsilon_2}\frac{1}{r^2}$$

and for $r \rangle r_2$

$$E(r) = \frac{Q}{4\pi\varepsilon_1}\frac{1}{r^2}$$

c). Potential:

for $r \le r_1$:

$$\varphi(r) = \int_r^\infty E\,dr = \int_r^{r_1} E\,dr + \int_{r_1}^{r_2} E\,dr + \int_{r_2}^\infty E\,dr = 0 + \frac{Q}{4\pi\varepsilon_2}\int_{r_1}^{r_2}\frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon_1}\int_{r_2}^{\infty}\frac{dr}{r^2}$$

After a little rearrangement we obtain

$$\varphi\,(r) = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1\, r_2} + \frac{1}{\varepsilon_2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right] = const.$$

For $r_1 \le r \le r_2$:

$$\varphi\,(r) = \frac{Q}{4\pi\varepsilon_2}\int_r^{r_2}\frac{dr}{r^2} + \frac{Q}{4\pi\varepsilon_1}\int_{r_2}^{\infty}\frac{dr}{r^2}$$

or after a little rearrangement we have

$$\varphi\,(r) = \frac{Q}{4\pi}\left[\frac{1}{r_2}\left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right) + \frac{1}{\varepsilon_2\, r}\right]$$

For $r \rangle r_2$ we have $\varphi\,(r) = \dfrac{Q}{4\pi\varepsilon_1}\displaystyle\int_r^{\infty}\frac{dr}{r^2}$

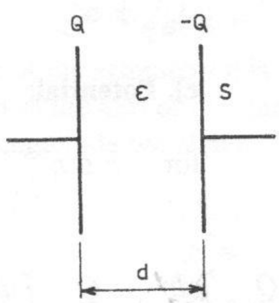
which we can rearrange as

$$\varphi\,(r) = \frac{Q}{4\pi\varepsilon_1}\,\frac{1}{r}$$

The dependencies of electric displacement, electric field and potential on the distance from the centre of the sphere are shown in the previous figures.

Problem 3-31. Determine a formula for the capacitance of a parallel-plate capacitor. The capacitor is charged so that the first plate has a charge $+Q$ and the second plate has a charge $-Q$. The surface charge density is σ.

<u>Solution:</u> Each plate of a capacitor has an area S and the two plates are separated by a distance d. We assume that d is small compared to the dimensions of each plate, so that the electric field is uniform between them. As long as the surface charge density is σ, the total charge is $Q = \sigma.S$. We saw earlier (problem 3-25) that the electric field between two closely spaced parallel plates has the magnitude $E = \dfrac{\sigma}{\varepsilon}$.



The voltage in the homogeneous electric field is

$$U = E\,d = \frac{\sigma}{\varepsilon}\,d$$

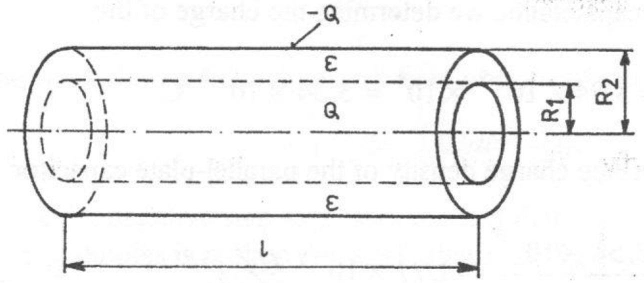
Thus, for the capacitance in terms of the geometry of the plates we obtain

$$C = \frac{Q}{U} = \varepsilon\,\frac{S}{d}$$

Problem 3-32. Determine a formula for the capacitance of a cylindrical capacitor. This capacitor consists of two coaxial cylinders of radii R_1 and R_2 and length l. Assume that the capacitor is very long (that is $l >> R_2$) so that we can ignore the fringing of the field lines at the ends for the purpose of calculating the capacitance. The capacitor is charged so that one cylinder has a charge $+Q$ (say, the inner one) and the outer one has a charge $-Q$.

Solution: We need to determine the voltage between the cylinders in terms of Q. To do this we determine the electric field using Gauss' law. As a Gaussian surface we construct a coaxial cylinder of radius r ($R_1 < r < R_2$) and length l. No flux passes through the end caps of the cylinders, so it all passes through the curved surface. Since E is uniform over this surface, whose area is $2\pi r\, l$, Gauss' law gives

$$\oint\!\!\oint_{(S)} \mathbf{E}\, d\mathbf{S} = E \oint\!\!\oint_{(S)} dS_n = E\, 2\pi r\, l = \frac{Q}{\varepsilon} \qquad \text{or} \qquad E = \frac{1}{2\pi\varepsilon\, l}\frac{Q}{r}$$



The voltage between the cylinders is

$$U = \int_{R_1}^{R_2} \mathbf{E}\, d\mathbf{r} = \frac{Q}{2\pi\varepsilon\, l}\int_{R_1}^{R_2}\frac{dr}{r} = \frac{Q}{2\pi\varepsilon\, l}\ln\frac{R_2}{R_1}$$

Finally the capacitance is

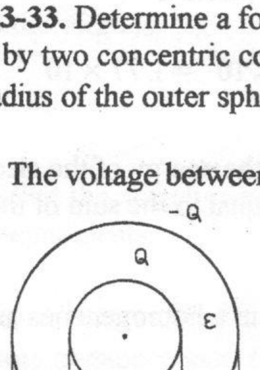
$$C = \frac{Q}{U} = \frac{2\pi\varepsilon\, l}{\ln\dfrac{R_2}{R_1}}$$

We can also express the capacitance per unit of length of a coaxial conductor as

$$c = \frac{C}{l} = \frac{2\pi\varepsilon}{\ln\dfrac{R_2}{R_1}}$$

Problem 3-33. Determine a formula for the capacitance of a spherical capacitor which is formed by two concentric conducting spheres. The radius of the inner sphere is R_1 and the radius of the outer sphere is R_2. The charge of the capacitor is Q.

Solution: The voltage between the spheres is



$$U = \int_{R_1}^{R_2} E\, dr = \frac{Q}{4\pi\varepsilon}\int_{R_1}^{R_2}\frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon}\frac{R_2 - R_1}{R_1 R_2}$$

Thus, for capacitance we have

$$C = 4\pi\varepsilon\frac{R_1 R_2}{R_2 - R_1}$$

Problem 3-34. The distance between the plates of the parallel-plate capacitor is 5×10^{-3} m. The area of each of the plates is 2 m^2. The potential difference between the plates of the capacitor, which is placed in a vacuum, is 10^4 V. Determine:

a) the capacitance of the capacitor,
b) the charge on each of the plates,
c) the surface charge density,
d) the electric field between the plates,
e) the electric displacement between the plates.

Solution:

a) The capacitance of a parallel-plate capacitor with a vacuum as a dielectric is

$$C = \varepsilon_0 \frac{S}{d}$$

After substituting we obtain

$$C = \frac{8.85 \times 10^{-12} \times 2}{5 \times 10^{-3}} = 3.54 \times 10^{-9} \; F = 3.54 \; nF$$

b) From the expression for capacitance we determine the charge of the capacitor

$$Q = C.U = 3.54 \times 10^{-9} \times 10^{4} = 3.54 \times 10^{-5} \; C$$

c) We can determine the surface charge density of the parallel-plate capacitor as

$$\sigma = \frac{Q}{S} = \frac{3.54 \times 10^{-5}}{2} = 1.77 \times 10^{-5} \; C/m^2$$

d) The electric field between the plates of the parallel-plate capacitor is

$$E = \frac{\sigma}{\varepsilon_0} = \frac{1.77 \times 10^{-5}}{8.85 \times 10^{-12}} = 2 \times 10^{6} \; V/m$$

We can obtain the same result using the formula

$$E = \frac{U}{d} = \frac{10^4}{5 \times 10^{-3}} = 2 \times 10^{6} \; V/m$$

e) Finally the electric displacement between the plates is

$$D = \varepsilon_0 \, E = 8.85 \times 10^{-12} \times 2 \times 10^{6} = 1.77 \times 10^{-5} \; C/m^2$$

Problem 3-35. Using the expression for the density of the energy of the electric field show that the magnitude of the polarisation vector is equal to the sum of the dipole moments per unit of volume of dielectric material.

Solution: The density of the energy of the electric field in a homogeneous and isotropic dielectric material is

$$w_D = \frac{1}{2} \varepsilon_0 \, \varepsilon_r \, E^2$$

The density of the energy of the electric field in a vacuum is

$$w_D = \frac{1}{2}\varepsilon_0 E^2$$

The difference in the density of the energy of the electric field in dielectric material and in a vacuum is

$$w_D - w_0 = \frac{1}{2}\varepsilon_0(\varepsilon_r - 1)E^2 = \frac{1}{2}\varepsilon_0 \kappa E^2 \qquad (1)$$

where $\kappa = \varepsilon_r - 1$ is the electric susceptibility.

In order to solve our problem we must consider the time-average distribution of the electrons on their orbits as a kind of spherically symmetric electron cloud enveloping the nucleus. The centre of the negative charge of the cloud in the absence of an applied field is coincident with the centre of the positive charge. When a field is applied, the cloud is shifted by a distance l. Hence an electric dipole originates, with the electric dipole moment $p = Q\,l$.

Let us suppose that the force F causing the displacement of the electron cloud with respect to the nucleus is proportional to the displacement l or $F = -K\,l$, where K is the proportionality constant. This force can also be expressed as $F = Q\,E$. From the balance of these two forces we can express the constant K as

$$K = \frac{QE}{l} \qquad (2)$$

The work that must be done to create one electric dipole is

$$A = \int_0^l F\,dl = -\int_0^l K l\,dl = -\frac{1}{2}Kl^2 = -\frac{1}{2}\frac{Q^2E^2}{K}$$

where the negative sign expresses the fact that the energy is used. If the volume density of the electric dipoles is n, then the work that must be done to form them also expresses the density of the energy per unit of volume of dielectric material, or

$$nA = \frac{nQ^2E^2}{2K} \qquad (3)$$

From comparison of equations (1) and (3) we obtain

$$\frac{nQ^2E^2}{2K} = \frac{1}{2}\varepsilon_0 \kappa E^2$$

which we can rewrite as

$$\kappa\varepsilon_0 = \frac{nQ^2}{K} \qquad (4)$$

The polarisation vector is defined as $\mathbf{P} = \varepsilon_0 \kappa \mathbf{E}$. The magnitude of this vector with respect to equation (4) therefore is

$$P = \frac{nQ^2E}{K} = \frac{nQ^2El}{QE} = nQl = np$$

Thus we see that the magnitude of the polarisation vector is equal to the volume density of the electric dipole moments.

Problem 3-36. Charges $Q_1 = 300\times10^{-9}\,C$ and $Q_2 = 750\times10^{-9}\,C$ are placed on two parallel-plate capacitors of capacitance $C_1 = 100\ pF$ and $C_2 = 50\ pF$, respectively. The upper plates have charges of opposite polarity.

1) Determine the voltage on each of the capacitors.
2) What happens if only the lower plates are connected together?
3) What happens if the upper plates are also connected together?

4) Determine the decrease in the energy stored in the capacitors and explain what happened with this energy.

Solution:

1): $U_1 = \dfrac{Q_1}{C_1}$ $\qquad U_2 = \dfrac{Q_2}{C_2}$

$U_1 = 3\times 10^3\ V$ $\qquad U_2 = 15\times 10^3\ V$

2) Nothing will happen. The charges on the lower plate are bounded by the charges on the upper plate, which cannot move.

3) In this case we have two capacitors connected in parallel. The charges are partially balanced. The total charge on both capacitors (for example on the upper plates) is

$$Q' = -Q_1 + Q_2 = 4.5\times 10^{-7}\ C$$

$$C' = C_1 + C_2 = 150\times 10^{-12}\ F$$

The voltage U' on both connected capacitors is

$$U' = \frac{Q'}{C'} = \frac{4.5\times 10^{-7}}{150\times 10^{-12}} = 3\times 10^3\ V$$

$$Q_1' = C_1 U' = 1\times 10^{-10}\times 3\times 10^3 = 3\times 10^{-7}\ C$$

$$Q_2' = C_2 U' = 5\times 10^{-11}\times 3\times 10^3 = 1.5\times 10^{-7}\ C$$

4) The energy of the capacitors before their connection is

$$W_1 = \frac{1}{2}C_1 U_1^2 = \frac{1}{2}1\times 10^{-10}\times 9\times 10^6 = 4.5\times 10^{-4}\ J$$

$$W_2 = \frac{1}{2}C_2 U_2^2 = \frac{1}{2}5\times 10^{-11}\times 225\times 10^6 = 5.62\times 10^{-3}\ J$$

The energy of the two capacitors is therefore

$$W = W_1 + W_2 = 6.07\times 10^{-3}\ J$$

The total energy of the system of two capacitors connected in parallel is

$$W' = \frac{1}{2}(C_1 + C_2)U'^2 = \frac{1}{2}1.5\times 10^{-10}\times 9\times 10^6 = 6.75\times 10^{-4}\ J$$

The decrease of the energy is

$$\Delta W = W - W' = (60.75 - 6.75)\times 10^{-4} = 5.4\times 10^{-3}\ J$$

This energy is dissipated as the heating of the connecting wire, a flash of light, electromagnetic radiation, sound etc.

Problem 3-37. Determine the electrostatic potential energy contained in the electric field around a spherical conductor with a charge Q. The radius of the sphere, which is surrounded with a homogeneous dielectric material with permittivity ε, is R.

Solution: The total electrostatic potential energy contained in any electric field is

$$W=\frac{1}{2}\iiint_V \varepsilon E^2 dV$$

where the integration is performed over all charge -free space in which the field exists - that is, over all space outside the sources that produce the field. As long as the charge is uniformly distributed on the surface of the sphere, the electric field around this sphere is the same as the electric field of the point charge, which is placed in the centre of the sphere:

$$E = \frac{Q}{4\pi\,\varepsilon\, r^2}$$

With respect to spherical symmetry, it is more suitable to solve the problem in spherical co-ordinates. In this case the elementary volume is expressed as

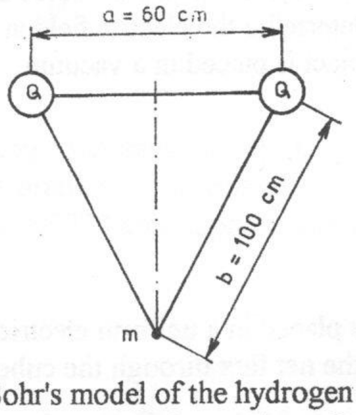
$$dV = r^2 \cos\varphi\, dr\, d\varphi\, d\vartheta$$

and the energy contained in the field is

$$W=\frac{1}{2}\frac{Q^2}{16\pi^2\varepsilon}\int_{r=R}^{\infty}\int_{\varphi=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\vartheta=0}^{2\pi}\frac{1}{r^2}\cos\varphi\, dr\, d\varphi\, d\vartheta=\frac{1}{8\pi}\frac{Q^2}{\varepsilon R}$$

Note: The same result can also be obtained from the expression $W = \dfrac{Q^2}{2C}$ where for C we substitute the capacitance of a single conducting sphere.

Problem 3-38. Two identical balloons filled with helium carry a small weight of mass $m = 5\,g$ (see the figure). The system is in equilibrium. Each of the balloons has the same charge Q. Determine this electric charge.



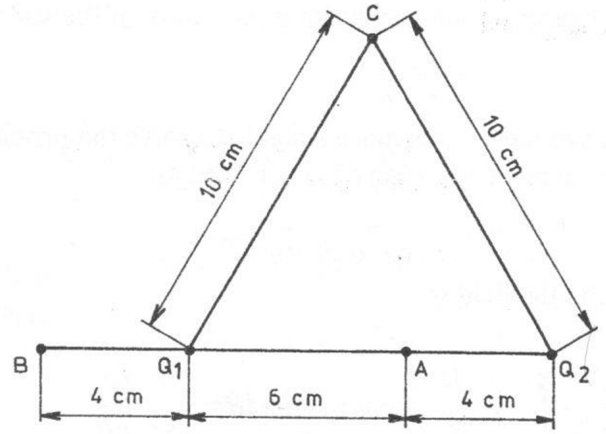
$$\left[\; Q=\sqrt{mg\frac{2a^3\pi\varepsilon_0}{\sqrt{b^2-\dfrac{a^2}{4}}}}=6\times10^{-7}C \;\right]$$

Problem 3-39. In the Bohr's model of the hydrogen atom the electron revolves in a circular orbit around the proton. Determine the number of revolutions of the electron per 1 s if the radius of the first circular orbit is $r = 5.28\times 10^{-11}\ m$.

$$\left[\; f=\frac{e}{2\pi\sqrt{4\pi^3\varepsilon_0 r^3 m_e}}=6.6\times10^{15}\,s^{-1} \;\right]$$

Problem 3-40. Two charges $Q_1 = 12\times 10^{-9}\ C$ and $Q_2 = -12\times 10^{-9}\ C$ are 10 cm apart. Determine the electric field at points A, B and C (see the following figure). Suppose that $\varepsilon_r = 1$.

$\left[E_A = 9.7\times10^4\ {}^{V}\!/_{m}\right]$ $\left[E_B = 6.2\times10^4\ {}^{V}\!/_{m}\right]$ $\left[E_C = 1.08\times10^4\ {}^{V}\!/_{m}\right]$



Problem 3-41. A point charge $+Q$ is placed at a distance a from the second point charge $+9Q$. Determine the co-ordinate x of the point between the charges in which the resultant electric field is equal to zero.

$[x = 0.25\ a]$

Problem 3-42. A thin, ring-shaped object of radius R holds a total charge $+Q$ distributed uniformly around it. Determine the electric field at the point on its axis, a distance r from the centre. The object is placed in a vacuum.

$$\left[E = \frac{Q\,r}{4\pi\varepsilon_0\left(R^2+r^2\right)^{\frac{3}{2}}}\right]$$

Problem 3-43. A cube of side l is placed in a uniform electric field E with edges parallel to the field lines. What is the net flux through the cube? What is the net flux through each of its six faces?

$\left[zero,\ E\,l^2,\ -E\,l^2,\ 0,\ 0,\ 0,\ 0\right]$

Problem 3-44. Find the work done by the field in moving an electric charge of q' in the field of a charge q (at the origin) from $(x, y, z) = (-1, 2, -3)$ to $(2, -1, -4)$.

$$\left[A = 0.049\ \frac{q\,q'}{4\pi\varepsilon_0}\right]$$

Problem 3-45. Derive an expression for the electrostatic potential energy of a charged particle q in a uniform (constant) electric field $\mathbf{E} = E_0\,\mathbf{i}$.

$$\left[\Delta W = q\,E_0\,(x_0 - x)\right]$$

Problem 3-46. Compute the electrostatic potential energy of a sphere which possesses an electric charge Q. The radius of the sphere is R, the potential of the sphere with respect to infinity is φ.

$$\left[W = \frac{1}{2} Q\,\varphi\right]$$

Problem 3-47. A very long straight wire possesses a uniformly distributed electric charge Q. The linear charge density is ξ. Determine the potential φ_r at the distance r from the wire.

$$\left[\varphi_r = -\frac{\xi}{2\pi\,\varepsilon_0}\ln|r| + C\right]$$

Problem 3-48. The electric potential of a very large metal plate is expressed by the formula $\varphi = \dfrac{a\cos\alpha}{r^2} + \dfrac{b}{r}$ where r and α are polar co-ordinates. Determine components E_r and E_α of the electric field at any point.

$$\left[E_r = \frac{1}{r^2}\left(b + \frac{2a\cos\alpha}{r}\right)\right] \qquad \left[E_\alpha = \frac{a\sin\alpha}{r^3}\right]$$

Problem 3-49. Determine the total electric charge Q of the Earth and its surface charge density σ if the gradient of potential at the surface of the Earth is 100 V/m and the radius of the Earth is 6378 km. Suppose that $\varepsilon_r = 1$.

$$\left[Q \approx 4.5 \times 10^5\ C\right] \qquad \left[\sigma = 8.8 \times 10^{-10}\ C/m^2\right]$$

Problem 3-50. A point charge $Q = -10^{-6}\,C$ is located at point (-1; 0; 0) of the orthogonal reference frame. The positive charge of equal magnitude is placed at point (1; 0; 0). Calculate the electric flux through the circle lying in the plane y,z with the centre in the origin of the reference frame. The radius R of the circle is 1m.

$$\left[\Phi_E = \frac{Q}{\varepsilon_0}\left(1 - \frac{a}{\left(R^2 + a^2\right)^{\frac{1}{2}}}\right) = 3.3 \times 10^4\ V.m\right]$$

Problem 3-51 The distance between the carbon [C] and oxygen [O] atoms in the group C=O which occurs in many organic molecules is $1.2\times10^{-10}\,m$ and the dipole moment of this group is $p=8\times10^{-30}\,C.m$.

Calculate:

a) the effective net charge on the positive carbon and the negative oxygen atoms,

b) the potential at the point the co-ordinates of which are: $r=9\times10^{-10}\,m$ and $\alpha=180°$ (see the figure of problem 3-28).

$\left[Q=6.6\times10^{-20}\,C\right]$ $\left[\varphi=-0.088V\right]$

Problem 3-52. The relative permittivity of helium at a temperature of 0° C and a pressure 0,1 MPa is 1.000074. Determine the electric dipole moment of the helium atom in the homogeneous electric field $E = 1\,V.m^{-1}$.

$\left[p= 2.4\times 10^{-37}\ C.m\right]$

Problem 3-53. Two parallel plates of area 1 m^2 are each given equal but opposite charges 30 μC. The space between them is filled with a dielectric material of relative permittivity $\varepsilon_r = 1{,}7$.

Determine:

a) the electric field E in the dielectric material,

b) the induced surface charge σ_i on each dielectric surface,

c) the electric field E_0 in the dielectric material due to the free charge,

d) the electric field E_i due to the induced surface charge.

$\left[E = 2\times 10^{6}\,V/m\right]$ $\left[\sigma_i=12.4\times10^{-6}\,C/m^2\right]$

$\left[E_0 = 3.39\times 10^{6}\,V/m\right]$ $\left[E_i = 1.39\times 10^{6}\,V/m\right]$

Problem 3-54. A parallel plate air capacitor has plates of area $S = 3\times 10^{-2}\,m^2$ and a separation of $h_1 = 3\ mm$. Between the plates, isolated form the ground, is placed another metal plate of the same area and thickness $h_2 = 1\ mm$. A battery charges the capacitor to a potential difference of 600 V and is then disconnected. The metal plate is then removed. Determine the work that must be done to remove the plate.

$\left[A=\dfrac{\varepsilon S U^2}{8h_2}\approx 12\times10^{-6}\,J\right]$

Problem 3-55. The plates of a parallel-plate capacitor of capacitance C which is charged on a potential difference U attract each other with a certain force. Derive the formula for the magnitude of this force.

$\left[F=\dfrac{1}{2}\dfrac{U^2 C}{x}\right]$