Problem 3-69. A battery of electromotive force $U_e = 1.06\,V$ and internal resistance $R_i = 1.8\ \Omega$ has a coil of resistance $R = 6.0\ \Omega$ connected across its terminals.
a) Find the potential difference U between the battery terminals.
b) Find the current in the circuit.
c) Find the dissipated power.

$[U = 0.815\ V]$ $[I = 0.136\ A]$ $[P = 33.2\ mW]$

Problem 3-70. The resistance of an immersion heater coil is 16 Ω. Calculate the time in which 400 g of water in the pot begins to boil if the initial temperature of water is 10° C and the efficiency of the heater is 60 %. The heater is designed to operate from 120 V. ($c_w = 4.18 \times 10^3\ J.kg^{-1}\ K^{-1}$).

$[\tau = 418\ s]$

Problem 3-71. The capacitor of a capacitance C = 100 μF and resistance of a dielectric material R = 10^6 Ω is connected to the source the voltage of which can be arbitrarily varied.
a) At voltage $U_1 = 100\ V$ calculate the displacement and the conduction current.
b) Assume that the voltage linearly increases from initial value $U_1 = 100\ V$ with a slope k = 10 V/s. Calculate the conduction and the displacement current after 20 s.

$[a)\ \ I_d = 0\ A \qquad I_c = 10^{-4}\ A]$

$[b)\ I_c = 3 \times 10^{-4}\ A \qquad I_d = 10^{-3}\ A]$

3.4 MAGNETIC FIELD

The magnetic field is associated with moving electric charges.
The basic magnetic field vector **B**, which is called **magnetic induction,** is operationally defined by the force acting on a moving electric charge q in a magnetic field as

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

where **v** denotes the velocity of the moving charge.
The magnetic field exerts a **force on a current carrying conductor** placed in this magnetic field

$$\mathbf{F} = \int_l (I.d\mathbf{l} \times \mathbf{B})$$

where **dl** is a vector in the direction of the current with a magnitude equal to the length of the current carrying element.

Biot-Savart's law is useful for determining the magnetic field due to a known arrangement of currents. It states that

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I}{r^2} (d\mathbf{l} \times \mathbf{r}_0)$$

where μ_0 is the permeability of a free space, $\mathbf{r}_0$ is a unit vector of the displacement vector pointing from **dl** to the point at which we determine magnetic induction.
To determine the orientation of the magnetic induction **B** we use **the right-hand rule**:

> If the current element is grasped by the right hand with the thumb pointing in the direction of current flow, then the curved fingers point in the direction of the magnetic field lines.

The magnetic flux Φ_m through the surface S is defined as

$$\Phi_m = \iint_S \mathbf{B}.d\mathbf{S}$$

The magnetic flux through a closed surface is equal to zero, or

$$\oiint_S \mathbf{B}.d\mathbf{S} = 0$$

Any current loop (circular or non circular) at distances that are sufficiently large appears to be a magnetic dipole. Any loop carrying a current I placed in a magnetic field **B** experiences a torque τ_m where

$$\tau_m = \boldsymbol{\mu} \times \mathbf{B}$$

where $\boldsymbol{\mu}$ is the magnetic dipole moment of the loop, which is defined as

$$\boldsymbol{\mu} = I.S.\boldsymbol{n}$$

In this expression **n** is a unit vector in the direction of the positive normal toward the surface S.
Potential energy of the magnetic dipole in the magnetic field is

$$U = -\boldsymbol{\mu}.\boldsymbol{B}$$

Ampere's law is stated for a circulation around a simple closed path L in space

$$\oint_L \mathbf{B}.d\mathbf{l} = \mu_0 I$$

where I is the net current passing through any surface bounded by the circulation loop L.
The magnetisation vector M is defined as the magnetic dipole moment per unit volume or

$$\mathbf{M} = d\boldsymbol{\mu}/dV.$$

The general definition of **magnetic field strength H** is

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

Ampere's law, which holds in the presence of magnetic materials, has the following form

$$\oint_L \mathbf{H}.d\mathbf{l} = I$$

For a linear and isotropic magnetic medium the magnetic induction can be expressed as

$$\mathbf{B} = \mu\,\mathbf{H} = \mu_0\,\mu_r\,\mathbf{H}$$

where μ is the permeability of the medium and μ_r is the relative permeability of the medium.

Magnetic susceptibility is defined as

$$\kappa_m = \mu_r - 1$$

Magnetic substances can be divided into three categories:

diamagnetic:	$\kappa_m < 0$	$\mu_r < 1$
paramagnetic:	$\kappa_m > 0$	$\mu_r > 1$
ferromagnetic:	$\kappa_m > 0$	$\mu_r \gg 1$

Faraday's induction law states that any change in total magnetic flux through a loop (whether due to the relative motion of the magnetic source and the loop or due to a change in the field) induces an electromotive force in the loop:

$$\oint_L \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\iint_S \mathbf{B}.d\mathbf{S}$$

where the minus sign is required by **Lenz's law,** which states that:

the induced electromotive force always acts to oppose the external change that generates it.

Self inductance is the constant of proportionality between the current and the magnetic flux

$$\Phi_B = L.I$$

The electromotive force U_e induced in a loop of inductance L when the current changes is

$$U_e = -L\frac{dI}{dt}$$

When there are two loops placed near each other, then the changing current I_1 in the first loop produces an electromotive force U_{e2} in the second loop, or

$$U_{e2} = -M\frac{dI_1}{dt}$$

where M is the mutual inductance. Mutual inductance is a constant of proportionality between the magnetic flux passing through the second loop and the current of the first loop (and vice versa)

$$M = \frac{\Phi_{21}}{I_1}$$

The energy stored in the magnetic field of current carrying conductor of inductance L is

$$W_m = \frac{1}{2}LI^2$$

The density of the energy stored in the magnetic field is

$$w_m = \frac{1}{2}\mathbf{H}.\mathbf{B}$$

The force between two parallel wires of the length l carrying electric currents I_1 and I_2 in the same direction is

$$F = \frac{\mu I_1 I_2 l}{2\pi d}$$

Problem 3-72. An electron which is accelerated by a voltage of 10 kV enters a uniform magnetic field of induction B = 1 T at right angles to the field. Describe the path of that electron.

Solution: The force on a single electron moving in the magnetic field is

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

The direction of the force is perpendicular to the field. The electron thus moves with constant speed in a circular path. The radius of the path is found using Newton's second law $\boldsymbol{F} = \boldsymbol{m.a}$, and the fact that the acceleration of a particle moving in a circle is $a = \frac{v^2}{r}$. Thus

$$\frac{m\,v^2}{r} = e.v.B$$

where e is a charge of an electron. We solve for r and find:

$$r = \frac{m.v}{e.B}$$

The velocity of an electron which is accelerated by a voltage of 10 kV can be found from its kinetic energy

$$eU = \frac{1}{2} m\,v^2$$

For velocity we obtain

$$v = \sqrt{\frac{2eU}{m}}$$

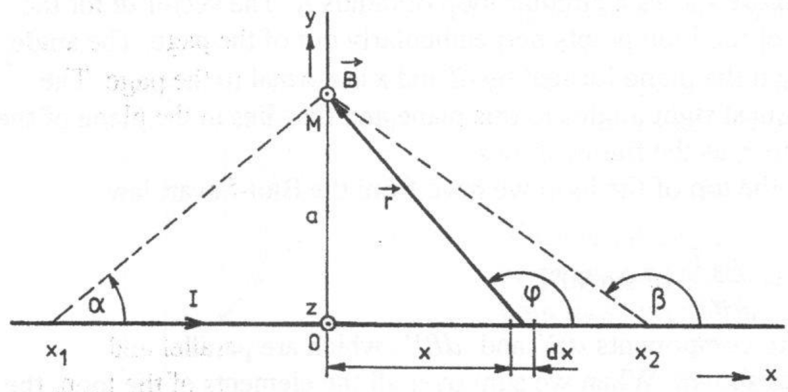
Substituting into the expression for radius we obtain

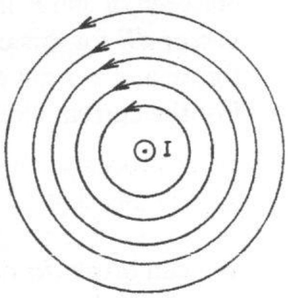
$$r = \frac{m}{eB}\sqrt{\frac{2eU}{m}} = \frac{9.1\times 10^{-31}}{1.6\times 10^{-10}\times 1}\sqrt{\frac{2\times 1.6\times 10^{-19}\times 10^{4}}{9.1\times 10^{-31}}} = 3.3\times 10^{-4}\ m$$

The relativistic effect of an increase in the mass of an electron has been disregarded.

Problem 3-73. From Biot-Savart's law derive an expression for magnetic induction at a distance r from a long straight wire carrying a current I.

Solution: To calculate the magnetic induction at a certain point in space due to a current flowing in a wire it is necessary first of all to calculate the contribution to the field at that point due to the current flowing in an infinitesimal part of the wire. Then, by means of integral calculus, we sum up the contributions due to all the infinitesimal elements in order to obtain the magnetic induction due to the whole wire.
As long as the magnetic field is axially symmetric about the wire, we consider the situation in two dimensions, as is shown in the figure, where the wire is assumed to stretch to infinity in both directions. Thus we can take the point M in which we calculate the magnetic induction to be on the y axis, which greatly simplifies the computation.





Consider an infinitesimal current element $I.dl = I.dx.i$ located at a distance x from the y axis. The displacement vector **r** from this current element to the field at point M is

$$r = -x.i + a.j$$

Thus the magnetic induction dB produced at point M by the current element is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I}{r^3}(dx.\mathbf{i} \times \mathbf{r})$$

which can be rewritten as

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I}{r^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx & 0 & 0 \\ -x & a & 0 \end{vmatrix} = \frac{\mu_0}{4\pi}\frac{I}{r^3}a.dx.\mathbf{k}$$

To compute the total magnetic induction at point M, we change the variable of integration, substituting

$$r = \frac{a}{\sin\varphi} \qquad x = -\,cotg\varphi \qquad dx = \frac{a.d\varphi}{\sin^2\varphi}$$

We obtain

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I}{a}\sin\varphi\, d\varphi.\mathbf{k}$$

Assuming that the length of the wire is $\Delta x = x_2 - x_1$ the magnetic induction at point M is calculated by integrating the previous equation between limits from α to β

$$\mathbf{B} = \mathbf{k}\frac{\mu_0 I}{4\pi a}\int_\alpha^\beta \sin\varphi\ d\varphi = -\,\mathbf{k}\frac{\mu_0 I}{4\pi a}\left[\cos\varphi\right]_\alpha^\beta = -\,\mathbf{k}\frac{\mu_0 I}{4\pi a}(\cos\beta\ - \cos\alpha\,)$$

For an infinitely long wire we have to integrate between limits from $\alpha = 0$ to $\beta = \pi$. After integration we obtain

$$\mathbf{B} = \frac{\mu_0 I}{2\pi a}\mathbf{k}$$

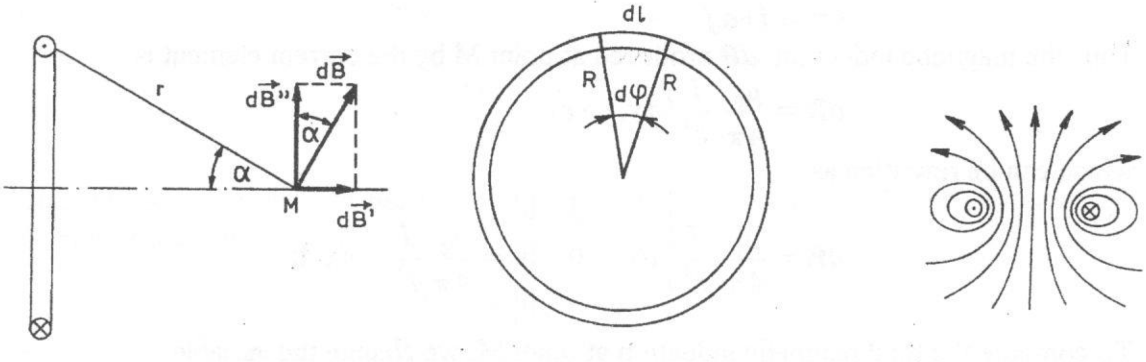
The lines of induction are also shown in the preceding figure. The magnetic induction at a given point is a vector tangent to the line of induction at this point. We determine the orientation of **B** using the right hand rule.

Problem 3-74. Determine the magnetic induction **B** for points on the axis of a circular loop of wire of radius R carrying a current I.

Solution: The following figure shows a circular loop of radius R. The vector $\boldsymbol{dl}$ for the current element at the top of the loop points perpendicularly out of the page. The angle between $\boldsymbol{dl}$ and $\boldsymbol{r}$ is 90°, and the plane formed by $\boldsymbol{dl}$ and $\boldsymbol{r}$ is normal to the page. The vector $\boldsymbol{dB}$ for this element is at right angles to this plane and thus lies in the plane of the figure and at right angles to $\boldsymbol{r}$, as the figure shows.
For the current element at the top of the loop we have from the Biot-Savart law

$$dB = \frac{\mu_0 I}{4\pi r^2} dl \sin 90°$$

We can break dB down into components dB' and dB'', which are parallel and perpendicular to the axis as shown. When we sum over all the elements of the loop, the perpendicular components will cancel out. Hence the total magnetic induction B will point along the axis and will be the sum of the contributions $dB' = dB \sin\alpha$.



Our figure shows that r and α are not independent of each other. For the integration let us use the substitutions

$$dl = r\,d\varphi \qquad \text{and} \qquad r = \frac{R}{\sin\alpha}$$

Thus, for the magnitude of magnetic induction at point M we can write

$$B = \frac{\mu_0 I \sin^3\alpha}{4\pi R}\int_0^{2\pi} d\varphi = \frac{\mu_0 I}{2R}\sin^3\alpha .$$

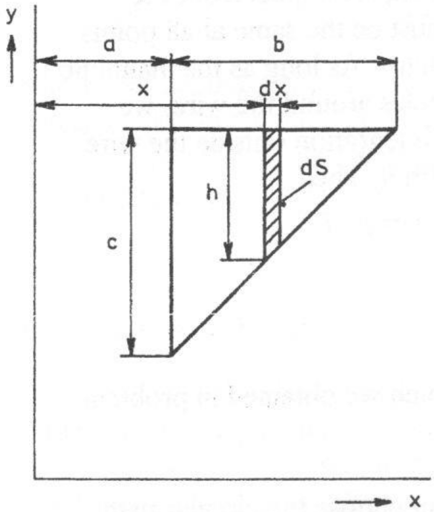
For the point at the centre of the loop, that is for $\alpha = \pi/2$ for magnetic induction we obtain

$$B = \mu_0 \frac{I}{2R}$$

The magnetic induction is directed along the axis of the loop and its orientation is found using the right hand rule.

Problem 3-75. Determine the magnetic flux through the loop in the shape of rectangular triangle, see the figure, where $a = 8\ cm$, $b = 10\ cm$ and $c = 10\ cm$. The loop is placed in the magnetic field. The lines of induction are perpendicular to the plane of the page. The magnetic induction as a function of the distance from the origin is expressed as $B = A/x$ where $A = 10^{-4}\ Wb/m$.

Solution: Since the magnetic induction in the area of the loop changes we determine the magnetic flux $d\Phi$ passing through the elementary area dS. In this area the magnetic induction can be considered to be a constant. Thus the magnetic flux through the area dS is



$$d\Phi = \mathbf{B}.d\mathbf{S}$$

Since the vector **B** and vector dS are both perpendicular to the plane of the triangle, that is both these vectors are parallel, we can write

$$d\Phi = \mathbf{B}.d\mathbf{S}\cos 0° = B\ dS$$

Substituting for $B = A/x$ and $dS = h.dx$ we have

$$d\Phi = \frac{A}{x} h\,dx$$

The quantity h can be found from the ratio

$$\frac{h}{(a+b-x)} = \frac{c}{b}$$

from which we have

$$h = \frac{c(a+b-x)}{b}$$

Thus for the magnetic flux through the elementary area dS we obtain

$$d\Phi = \frac{A}{x}\frac{c(a+b-x)}{b}dx.$$

The total magnetic flux Φ through the area of the triangle is

$$\Phi = \int_a^{a+b} \frac{A}{x}\frac{c(a+b-x)}{b}dx = \frac{Ac(a+b)}{b}\int_a^{a+b}\frac{dx}{x} - \frac{Ac}{b}\int_a^{a+b}dx =$$

$$= \frac{Ac(a+b)}{b}\ln\frac{(a+b)}{a} - \frac{Ac}{b}b = Ac\left(\frac{a+b}{b}\ln\frac{a+b}{b} - 1\right).$$

Substituting the numerical values we obtain

$$\Phi = 10^{-4} \times 0.1 \times \left[\frac{0.08+0.1}{0.1}\ln\frac{0.08+0.1}{0.1} - 1\right] = 0.45 \times 10^{-5}\ Wb$$

Problem 3-76. A long straight cylindrical wire conductor of radius R carries a current I of uniform current density in the conductor. Determine the magnetic induction at

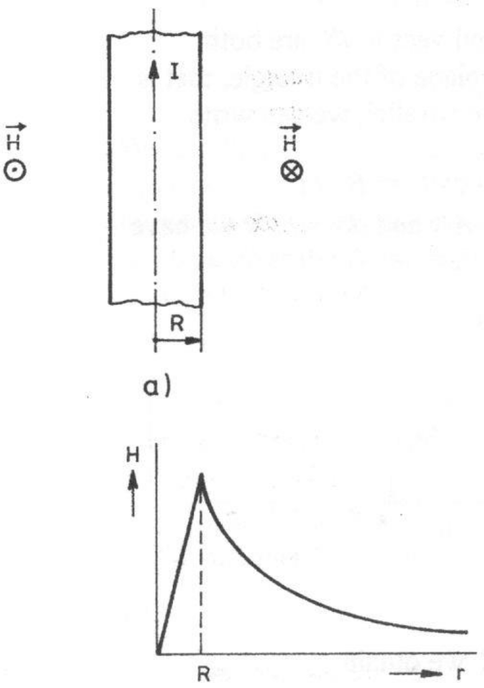
a) points outside the conductor ($r > R$) and

b) points inside the conductor ($r < R$).

Assume that r, the radial distance from the axis, is much less than the length of the wire.

Solution

a) Because the wire is long, straight and cylindrical, we expect from the symmetry of the situation that the magnetic induction must be the same at all points that are the same distance from the centre of the conductor. As long as the magnetic induction is tangent to circles around the wire, we choose a circular path of integration outside the wire ($r > R$), but concentric with it. Then



$$\oint \mathbf{B}\, d\,\mathbf{l} = B(2\pi\, r) = \mu_0\, I$$

or

$$B = \frac{\mu_0\, I}{2\pi r}$$

This is the same result which we obtained in problem 3-73

b) Inside the wire we again choose the circular path concentric with the cylinder, we expect **B** to be tangential to this path, and again, because of the symmetry, it will have the same magnitude at all pints on the circle. The current enclosed in this case is less than I by a factor of the ratio of the areas $\frac{\pi\, r^2}{\pi\, R^2}$. So Ampere's law gives

$$\oint \mathbf{B}\, d\,\mathbf{l} = B(2\pi\, r) = \mu_0\, I \left(\frac{\pi\, r^2}{\pi\, R^2}\right)$$

or

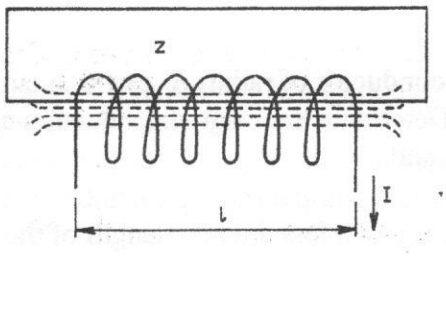
$$B = \frac{\mu_0\, I}{2\pi R^2}\, r$$

The field is zero at the centre of the conductor and increases linearly with r until $r = R$. Beyond $r = R$ magnetic induction decreases as $1/r$.

Problem 3-77. Determine the magnetic induction B inside a very long (ideally, infinitely long) closely packed solenoid carrying a current I, using Ampere's law. The length of the solenoid is l and the number of turns is N.

Solution: In order to apply Ampere's law we choose the integration path a,b,c,d which is shown in the following figure. We will consider this path as made up of four segments, the sides of the rectangle. Thus we can write Ampere's law as

$$\oint \mathbf{B}.d\mathbf{l} = \int_a^b \mathbf{B}.d\mathbf{l} + \int_b^c \mathbf{B}\, d\mathbf{l} + \int_c^d \mathbf{B}.d\mathbf{l} + \int_d^a \mathbf{B}.d\mathbf{l}$$



The magnetic field inside the solenoid is so small as to be negligible compared to the field inside; thus the first term will be zero. Furthermore, **B** is perpendicular to the segments bc and da inside the solenoid , and is nearly zero between and outside the coils.

Thus these terms are also zero. Therefore we have reduced the integral to

$$\oint \mathbf{B}.d\mathbf{l} = \int_c^d \mathbf{B}.d\mathbf{l} = B.l,$$

where B is the magnetic induction inside the solenoid and l is the length *cd.* Now we determine the current enclosed by this loop. If a current I flows in the wires of the solenoid, the total current enclosed by the integration path a,b,c,d is NI where N is the number of turns. Thus Ampere's law gives us

$$B.l = \mu_0 N I$$

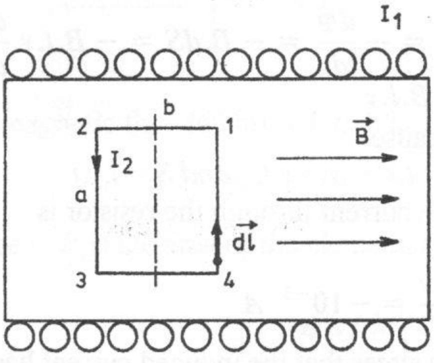
If we let $n = \frac{N}{l}$ be the number of turns per unit length, then

$$B = \mu_0 n I$$

Note that B depends only on the number of turns per unit length n, and the current I. The field does not depend on the position within the solenoid, so B is uniform. This is strictly true only for an infinite solenoid, but it is a good approximation for real solenoids for points not close to the ends.

Problem 3-78. A rectangular loop of sides $a = 5\ cm$ and $b = 3\ cm$ is placed inside an infinitely long solenoid which has $n = 1000$ turns per unit length and which carries a current of $I_1 = 1\ A$. The shorter sides of the loop are parallel with the axis of the solenoid, see the following figure. The loop carries a current $I_2 = 3\ A$. Calculate the torque of the force couple acting on the loop.

Solution: The expression for the magnetic induction in the area where the current carrying loop is placed was obtained in problem 3-77 as



$$B = \mu_0 n I_1$$

We determine the force acting on the different parts of the loop as

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}.$$

Due to the position of the loop with respect to the magnetic field the force acting on the sides *1,2* and *3,4* is equal to zero. The force acting on the side *4,1* which is $F_{4,1} = a.B.I_2$ (*dl* is perpendicular to ***B***) is equal but opposite to the force acting on side *2,3* which is $F_{2,3} = a.B.I_2$ (*dl* is perpendicular to ***B***). The torque of this force couple is $\tau = \boldsymbol{m} \times \boldsymbol{B}$ where $\boldsymbol{m} = I.S.\boldsymbol{n}$ is the magnetic dipole moment of the loop perpendicular to the plane of the loop.

In our case the magnetic induction ***B*** and the vector of the normal to the plane of the loop ***n*** are perpendicular to each other. Thus, for the magnitude of the torque of the force couple we can write

$$\tau = b.F = b.I_2.a.B = b.I_2.a\,\mu_0 I_1 n = 5.6 \times 10^{-6}\ N.m$$

Problem 3-79. The magnetic susceptibility of the monocrystal of Ge of the unit volume 1 m^3 is $\chi = -8\times10^{-6}$. Calculate the magnetic induction inside the crystal and the magnitude of the magnetisation vector if the magnetic field strength is $H = 1\times10^5\ A.m^{-1}$.

Solution: Magnetic induction as a function of magnetic field strength is expressed in the following way

$B = \mu_0\,\mu_r\,H = \mu_0(\chi+1)H = 4\pi\times10^{-7}\times\left(1-8\times10^{-6}\right)\times10^5 = 0.126\ T.$

The magnitude of the magnetisation vector $\boldsymbol{M}$ is

$$M = \mu_0\,\chi H = 4\pi\times10^{-7}\times\left(-8\times10^{-6}\right)\times10^5 = -1.005\times10^{-6}\ T.$$

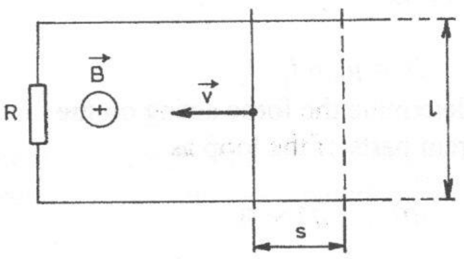
Note that due to the value of the magnetic susceptibility Ge is a diamagnetic material.

Problem 3-80. Suppose a conducting rod placed on two long frictionless and resistanceless parallel rails a distance $l = 0,1\ m$ apart in a uniform magnetic field $B = 0,1\ T$ perpendicular to the rails and rod, as shown in the following figure.

a) Calculate the current flowing through the conductor $R = 10\ \Omega$ if the rod moves at a constant speed $v = 1\ m/s$. (Disregard the magnetic field due to the current).

b) What will change the source of the magnetic field moves together with the conductor.

Solution:



a) Disregarding the magnetic field due to the current in the conductor the induced electromotive force is

$$U_\varepsilon = -\frac{d\Phi}{dt} = -B.dS = -B.l.v.\frac{dt}{dt} = = B.l.v$$

because

$S = l.s \quad s = v.t \quad and\ S = l.v.t$

The current through the resistor is determined from the Ohm's law or

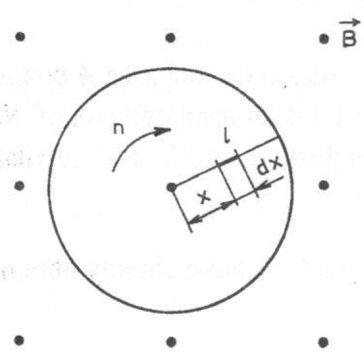
$$I = \frac{U_\varepsilon}{R} = -\frac{B.l.v}{R} = -10^{-3}\ A$$

The negative sign is required by Lenz's law. This states that the induced current has such orientation as to oppose the motion of the conductor in the magnetic field.

b) The current does not change because the magnetic flux through the loop will change with the rate as in the previous case.

Problem 3-81. A straight rod of length $l = 15\ cm$ rotates in a uniform magnetic field of induction $B = 0.5\ T$ in a plane perpendicular to the plane of rotation, see the following figure. The number of revolutions of the rod is $n = 60\ 1/s$. Find the electromotive force developed between the two ends of the rod.

Solution: The rod may be divided into elements of length dx. If a wire of length dx is moved at a velocity $\boldsymbol{v}$ at right angles to a field $\boldsymbol{B}$ an induced electromotive force is



given by the expression

$$dU_i = -B.v.dx$$

where the velocity of the element of length *dx* is $v = 2.\pi.x.n$. Substituting into the previous equation we have

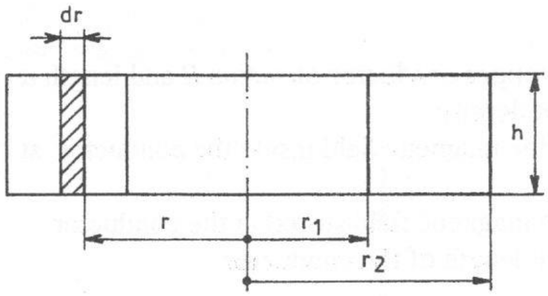
$$dU_i = -B.2\pi.x.n.dx.$$

Each element is perpendicular to ***B*** and also moves in a direction at right angles to ***B*** so that, since the dU_i are "in series", the total electromotive force induced in a wire is found by integration or

$$U_i = \int_0^l dU_i = -2\pi nB\int_0^l x.dx = -\pi n.B.l^2 = -2.12\ V.$$

Problem 3-82. Derive an expression for the inductance of a toroid of rectangular cross section as shown in the following figure. The number of turns is *n*.

Solution: The lines of induction for the toroid are concentric circles.



Applying Ampere's law

$$\oint \mathbf{H}.d\mathbf{l} = nI$$

to a circular path of radius *r* yields $H.2\pi r = n.I$. Solving for *H* yields

$$H = \frac{n.I}{2\pi\ r}.$$

The magnetic flux dΦ through the dashed cross section is

$$d\Phi = BdS = \frac{\mu\ nI}{2\pi\ r}h.dr$$

where *h.dr* is the area of the elementary strip. The magnetic flux through one turn is

$$\Phi = \frac{\mu\ nI\ h}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\mu\ nI\ h}{2\pi}\ln\frac{r_2}{r_1}$$

The total magnetic flux is $\Phi_{total} = n.\Phi$

Thus the inductance of the toroid is

$$L = \frac{\Phi_{total}}{I} = \frac{\mu\ n^2\ h}{2\ \pi}\ln\frac{r_2}{r_1}.$$

Problem 3-83. Calculate the electromotive force induced in a coil of inductance $L = .,06\ H$ when the current in a coil changes uniformly with a rate of *10 A/s.*

Solution: The self-induced emf in the coil of inductance *L* carrying a current *I* is

$$U_e = -L\frac{dI}{dt} \quad \text{or} \quad U_e = -L\frac{\Delta I}{\Delta t}$$

Substituting the numerical values we obtain $U_e = -0.6\ V$.

Problem 3-84. A long thin solenoid of length l and cross-sectional area A contains N_1 closely packed turns of wire. Wrapped tightly around it is an insulated coil of N_2 turns. Assume all the flux from coil 1 (the solenoid) passes through coil 2, and calculate the mutual inductance.

Solution: For the magnetic induction inside the solenoid we have already obtained (see problem 3-77)

$$B = \mu_0 \frac{N_1}{l} I_1$$

where I_1 is the current in the solenoid. Since all the flux in the solenoid links the coil, the flux $\Phi_{2,1}$ through coil 2 is

$$\Phi_{2,1} = B.A = \mu_0 \frac{N_1}{l} I_1 A$$

Hence the mutual inductance is

$$M = \frac{N_2 \Phi_{2,1}}{I_1} = \frac{\mu_0 N_1 N_2 A}{l}$$

Note that the mutual inductance M depends only on geometric factors, and not on the currents.

Problem 3-85. A long straight cylindrical copper conductor of radius R and length a carries a steady current I of uniform current density.

a) Determine the energy density of the magnetic field inside the conductor at distance r from its centre.
b) Determine the total energy of the magnetic field stored in the conductor.
c) Determine the inductance per unit length of the conductor.

Solution: For the magnetic induction inside a conductor carrying a current I we have already obtained (see problem 3-76):

$$B = \frac{\mu I}{2\pi R^2} r$$

a) The density of the energy stored in the magnetic field is

$$w_m = \frac{1}{2} \mathbf{H}.\mathbf{B}$$

Since in our case the medium is homogeneous and isotropic (the vectors ***B*** and ***H*** are co-linear) we can write

$$w_m = \frac{1}{2} B.H = \frac{B^2}{2\mu}$$

Substituting for B we obtain

$$w_m = \frac{1}{8} \frac{\mu I^2 r^2}{\pi^2 R^4}.$$

b) To determine the total energy of the magnetic field stored in the conductor of length a and carrying current I we first determine the energy dW stored in a volume element dV consisting of a cylindrical shell whose radii are r and dr and whose length is a.

Thus

$$dV = 2\pi radr \qquad \text{and} \qquad dW = w_m\, dV$$

The total stored magnetic energy in the conductor is found by integration

$$W_m = \iiint_V w_m dV = \int_0^R \frac{1}{8}\frac{\mu I^2 r^2}{\pi^2 R^4} 2\pi\, radr = \frac{\mu I^2 a}{4\pi R^4}\int_0^R r^3 dr = \frac{\mu I^2 a}{16\pi}.$$

c) We can find the inductance L of a current carrying conductor from the equation

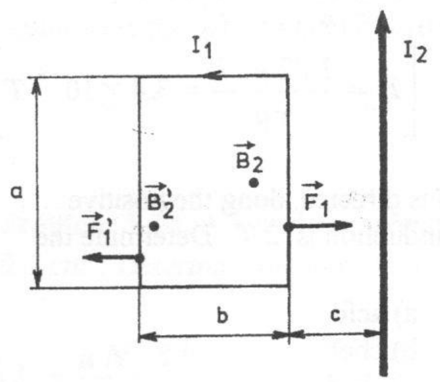
$$W_m = \frac{1}{2} L I^2$$

which leads to

$$L = \frac{2W_m}{I^2} = \frac{\mu a}{8\pi}.$$

Problem 3-86. A rectangular loop of wire of sides $b = 10\ cm$ and $a = 20\ cm$ carries a current $I_1 = 10\ A$ and lies in a plane which also contains a very long straight conductor carrying a current $I_2 = 20\ A$. Determine the net force on the loop due to the magnetic field of the straight conductor. The loop is placed in a vacuum.

Solution: The loop is placed in the magnetic field of the long straight conductor. Such a case was solved in problem 3-76. Thus for the side of the loop which is closer to the conductor the magnetic induction is $B_2 = \frac{\mu_0 I_2}{2\pi c}$



The magnetic induction for the more distant side of the loop is

$$B_2' = \frac{\mu_0 I_2}{2\pi(c+b)}$$

The force acting on the element of the current carrying conductor is

$$d\mathbf{F} = I(d\mathbf{l} \times \mathbf{B})$$

Since in our case $d\boldsymbol{l}$ is perpendicular to $\boldsymbol{B}$ we can write

$$dF = I\, dl\, B$$

Due to the fact that both the conductor and the loop are straight we can also write

$$F = B\, I\, l \sin\alpha$$

where l is the corresponding length. Thus the forces acting on the longer sides are

$$F_1 = I_1\, a\, B_2 \qquad \text{and} \qquad F_1' = I_1\, a\, B_2'.$$

The orientation of these forces, determined by the right hand rule, is shown in the figure. The forces acting on the shorter sides are of equal magnitudes but opposite orientations. Thus they cancel each other out.

The net force acting on the loop is therefore

$$F = F_1 - F_1' = I_1\, a(B_2 - B_2') = \frac{I_1 I_2\, a\, \mu_0}{2\pi}\left(\frac{1}{c} - \frac{1}{c+b}\right) = 1.06 \times 10^{-4}\ N$$

This net force acts in the plane of the loop.

Problem 3-87. An electron has a velocity $v = 10^4\ m\,s^{-1}$. It enters a uniform magnetic field of induction $B = 10^{-2}\ T$ in such a way that its path makes an angle $\alpha = 30°$ with a lines of induction. Describe its path.

[The electron will perform a spiral motion with a radius

$$R = \frac{m\,v\sin\alpha}{e\,B} = 2.84\times10^{-6}\ m$$

and a pitch (distance between loops)

$$h = 2\pi R\ cotg\ \alpha = 3.1\times10^{-5}\ m\ ]$$

Problem 3-88. A copper disc of radius $R = 20\ cm$ has an axis parallel with lines of induction of a uniform magnetic field of induction $B = 0.1\ T$. The current $I = 5\ A$ flows from the friction contact on the circumference of the disc to the friction contact on the axis of the disc. Calculate the torque acting on the disc.

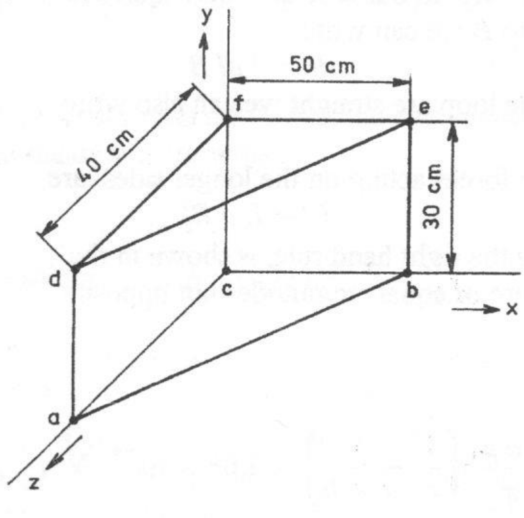
$$\left[\tau_m = \frac{1}{2} I R^2\ B = 0.01\ Nm\right]$$

Problem 3-89. The square loop of the side $a = 10\ cm$ carrying a current $I = 30\ A$ is placed in a vacuum. Calculate the magnetic induction at the centre of the loop.

$$\left[B = \frac{2\sqrt{2}\,\mu_0\,I}{\pi\,a} = 3.4\times10^{-4}\ T\right]$$

Problem 3-90. The vector of magnetic induction $\boldsymbol{B}$ is directed along the positive direction of the x axis. The magnitude of magnetic induction is $2\ T$. Determine the magnetic flux Φ_m through the plane:

a) acfd
b) cbef
c) abed.



$$[a)\quad \Phi_m = 0.24\ Wb]$$
$$[b)\quad \Phi_m = 0]$$
$$[c)\quad \Phi_m = 0.24\ Wb]$$

Problem 3-91. A long straight wire of circular cross section and radius *a* carries a current *I*. It is insulated from and surrounded by a conducting cylindrical shell of inner radius *b* and outer radius *c*. This shell carries an equal and opposite current, uniformly distributed over its cross section. Compute and sketch the magnetic field of the two conductors.

$$\left[r \le a \qquad H_1 = \frac{I\,r}{2\pi\,a^2}\right] \qquad \left[a \le r \le b \qquad H_2 = \frac{I}{2\pi\,r}\right]$$

$$\left[b \le r \le c \qquad H_3 = \frac{I}{2\pi r}\frac{c^2 - r^2}{c^2 - b^2}\right]$$

Problem 3-92. Liquid oxygen has a magnetic susceptibility of $\kappa_m = 0.0034$. Determine its permeability μ and relative permeability μ_r.

$$\left[\mu = 1.261 \times 10^{-6}\right] \qquad \left[\mu_r = 1.0034\right]$$

Problem 3-93. A square loop of area $S_r = 25\ cm^2$ is placed in a uniform magnetic field of induction $B = 0{,}2\ T$ so that the normal to the plane of the loop is parallel with magnetic induction **B**. The loop is made from copper wire of cross section $S_d = 2\ mm^2$. Calculate the charge passing through the loop when the magnetic field is switched off. $\left(\rho_{Cu} = 0.017 \times 10^{-6}\ \Omega.m\right)$

$$\left[Q = \frac{B\,S_d\,\sqrt{S_r}}{4\rho_{Cu}} = 0.29\ C\right]$$

Problem 3-94. A *5 cm* long solenoid has a total $N = 100$ turns and a cross section $S = 0.3\ cm^2$. Determine the inductance of the solenoid for air and iron core $\left(\mu_{r_{Fe}} = 4000\right)$.

$$\left[L = \frac{\mu N^2 S}{l}\right] \qquad \left[L_{air} = 7.5 \times 10^{-6}\ H\right] \qquad \left[L_{Fe} = 3 \times 10^{-2}\ H\right]$$

Problem 3-95. A current $I = 2\ mA$ produces a magnetic field in a coil of a length $l = 50\ cm$. The coil has 10 000 turns of radius $d = 6\ cm$. How much energy is stored in the magnetic field?

$$\left[W_m = \frac{1}{2} L I^2 = 1.42 \times 10^{-6}\ J\right]$$

Problem 3-96. Two long straight parallel conductors separated by a distance $d = 1\ m$ are carrying equal but opposite currents $I_1 = I_2 = 1\,A$. Determine the force acting per unit of length of each of the conductors.

$$\left[\frac{F}{l} = \frac{\mu_0\, I_1\, I_2}{2\pi d} = 2 \times 10^{-7}\ N.m^{-1}\right]$$