1. KINEMATICS

1 - 1 Velocity

Let us suppose that a particle moves along the x axis of a rectangular coordinate system and let Fig. 1-1 represent a graph of the position x of the particle versus time t .

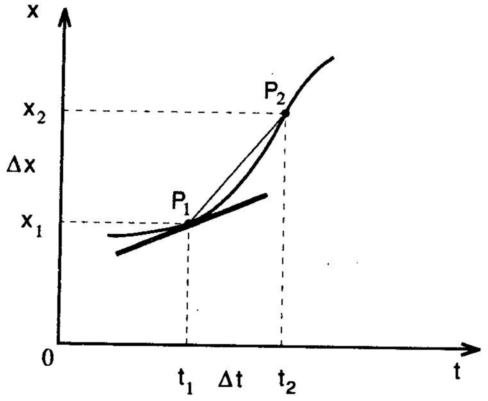

Figure 1 - 1

At time t_1 the particle is at position x_1 and at time t_2 at position x_2 . $P_1(x_1, t_1)$ and $P(x_2, t_2)$ represent these two points on the graph. The ratio $\Delta x/\Delta t$ is the slope of the straight line P_1P_2. This ratio is also the average velocity of the particle during time interval $\Delta t = t_2 - t_1$. We can conclude that the average velocity of an object during any time interval $\Delta t = t_2 - t_1$ is equal to the slope of the straight line connecting the two points (x_1, t_1) and (x_2, t_2) on an x vs. t graph.

Now we can give the definition of the instantaneous velocity at a given instant (say t_1, at which time the particle is at x_1) as the limiting value of the average velocity as Δt approaches zero. And we see that it equals the slope of the tangent to the x vs. t curve at that time (which we simply call "the slope of the curve" at that point).

Hence

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} . \qquad (1\text{-}1)$$

This limit is called the derivative of x with respect to t . Eq. (1-1) is the definition of the velocity for one-dimensional motion. We may say that the velocity of an object is the rate at which its displacement changes with time. The unit of the velocity is the unit of displacement divided by the unit of time. In the SI system it is meter per second - m/s.

If an object moves with constant velocity over a particular time interval the graph of x vs. t will be a straight line whose slope equals the velocity.

1 - 2 Acceleration

An object whose velocity is changing in time is said to be accelerating. Let us consider a graph of the velocity v vs. time t as shown in Fig. 1-2.

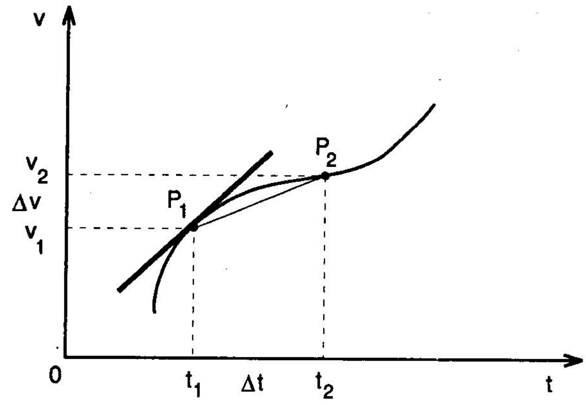

Figure 1 - 2

The average acceleration $\Delta v/\Delta t$ over a time interval $\Delta t = t_2 - t_1$ is represented by the slope of the straight line connecting two points P_1 and P_2.

The instantaneous acceleration at any time (say t_1) is equal to the slope of the tangent to the v vs. t curve at that time.

So, the instantaneous acceleration is defined as the limiting value of the average acceleration as Δt approaches zero.

Hence

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} . \qquad (1\text{-}2)$$

Here dv/dt is the derivative of v with respect to t and d^2x/dt^2 is called the second derivative of x with respect to time t . Eq. (1-2) is the definition of the acceleration for one-dimensional motion. We may say that the acceleration of an object is the rate at which its velocity changes with time.

The unit of the acceleration in the SI system is meter per second squared - m/s^2.

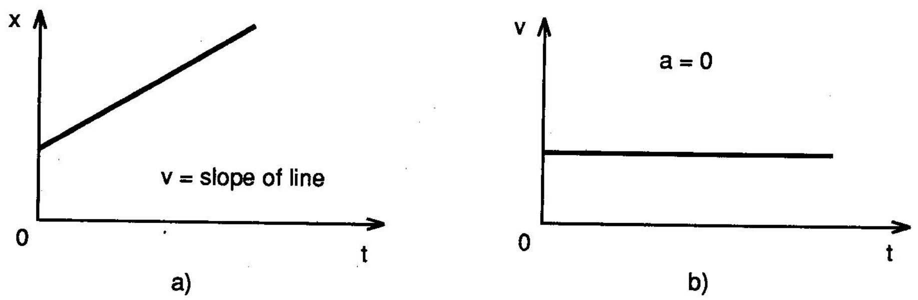

Figure 1 - 3

If the velocity of an object is constant then its acceleration equals zero, since $\Delta v = 0$. The x vs. t graph for constant velocity is now a straight line whose slope equals the magnitude of the velocity (Fig. 1-3a). The v vs. t graph in this case is also a straight line but its slope is zero, so the straight line is parallel to the t axis (Fig. 1-3b).

1 - 3 Uniformly Accelerated Motion

A uniformly accelerated motion occurs as the magnitude of the acceleration is constant and the motion is in a straight line. In this case the instantaneous and average acceleration are equal.

To simplify our notation we suppose that initial time equals zero, so, we put $t_1 = 0$. Let $t_2 = t$ be an elapsed time. The initial position (x_1) and initial velocity of an object are now represented by x_o and v_o respectively and at time t the position and velocity of an object will be called x and v respectively (rather than x_2 and v_2).

The average velocity during time t will be (it is signed with a bar over)

$$\bar{v} = \frac{x - x_o}{t} . \qquad (1-3)$$

Because the velocity increases at a uniform rate, the average velocity will also be midway between the initial and final velocities:

$$\bar{v} = \frac{v_o + v}{2} . \qquad (1-4)$$

The acceleration (which is assumed constant in time) will be

$$a = \frac{v - v_o}{t} . \qquad (1-5)$$

According to the definition of an acceleration given by Eq. (1-2) we have

$$v(t) = \int a\, dt + v_o = at + v_o \qquad (1-6)$$

where a is given acceleration which is constant and v_o is the initial velocity at time $t = 0$.

Now we can calculate the position of an object for any time t when it is moving with constant acceleration. From Eqs. (1-1) and (1-6) we obtain

$$x(t) = \int v\, dt = \int (at + v_o)\, dt = \frac{1}{2} at^2 + v_o t + x_o \qquad (1-7)$$

where x_o and v_o are initial position and initial velocity, respectively, of an object at time $t = 0$. Eqs. (1-6) and (1-7) are two useful ones of uniformly accelerated motion.

1 - 4 Displacement, Velocity and Acceleration Vectors

Suppose a particle follows a path in the xy plane as shown in Fig. 1-4. At time t_1 the particle is at point P_1 and at time t_2 it is at point P_2. The vector $\vec{r}_1$ is called the position vector of the particle at time t_1 and that $\vec{r}_2$ is the position vector at time t_2 (Any position vector represents the displacement

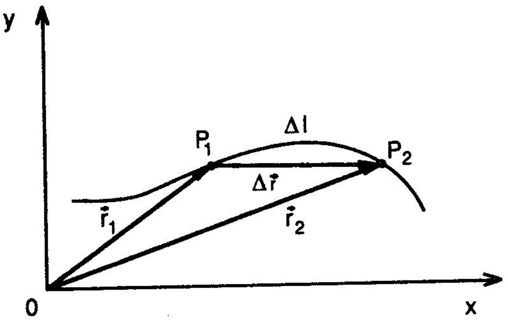

Figure 1 - 4

of the particle from the origin of the coordinate system.) The vector $\Delta\vec{r}$ is called the displacement vector. It is defined as the vector representing change in position of a particle:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \, .$$

This vector represents the displacement during the time interval $\Delta t = t_2 - t_1$.

In vector notation the both vectors can be expressed (in general , for three dimensions):

$$\vec{r}_1 = x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}$$

where x_1, y_1 and z_1 are coordinates of the point P_1 and $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors of unit length along the chosen coordinate axes (see appendix). Similarly,

$$\vec{r}_2 = x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k} \, .$$

Hence

$$\Delta\vec{r} = (x_2 - x_1)\vec{i} + (y_2 - y_1)\vec{j} + (z_2 - z_1)\vec{k} \, . \qquad (1\text{-}8)$$

The average velocity vector over the time interval $\Delta t = t_2 - t_1$ is defined as

$$\bar{\vec{v}} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\vec{r}_2 - \vec{r}_1}{t_2 - t_1}$$

where $\Delta\vec{r}$ is the change in the position vector during time interval Δt .

Note that the magnitude of the average velocity vector in Fig. 1-4 is not equal to the average speed which is the actual distance traveled $\Delta\ell$ divided by Δt. Only in case of a motion along a straight line in one direction, the average speed and the average velocity are equal. However, in the limit $\Delta t \to 0$, Δr always approaches $\Delta\ell$, so the instantaneous speed always equals the magnitude of the instantaneous velocity vector at any time.

Now let Δt approach zero so that the distance between points P_2 and P_1 approaches zero, too. We define the velocity vector (the instantaneous velocity vector) as the limit of the displacement vector as the time interval Δt is allowed to approach zero:

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} \, . \qquad (1\text{-}9)$$

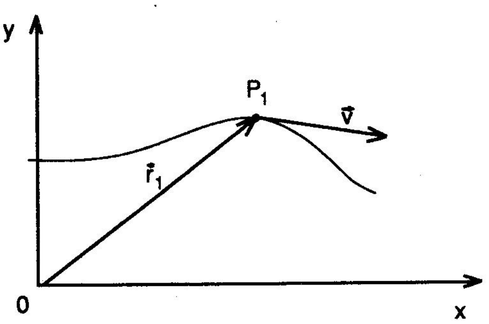

Figure 1 - 5

So, the velocity vector is defined as the derivative of the position vector with respect to time. It represents the rate of change of the position vector.

The direction of $\vec{v}$ at any moment (say, at P_1) is along the line tangent to the path at that moment (Fig. 1-5).

Eq. (1-9) can be written in terms of components as

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k} \qquad (1\text{-}10)$$

where

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt} \quad \text{and} \quad v_z = \frac{dz}{dt} \quad \text{are the } x, y \text{ and } z$$

components of the velocity vector, respectively, and $\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors that are constant in magnitude and direction. The magnitude of the velocity vector can be expressed as

$$v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} . \qquad (1\text{-}11)$$

In a similar way the average acceleration vector over a time interval Δt is defined as

$$\bar{\vec{a}} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

where $\Delta\vec{v}$ is the change in the instantaneous velocity during that time interval.

In general, $\vec{v}_2$ may not be in the same direction as $\vec{v}_1$. Then $\Delta\vec{v}$ may be in different direction from either $\vec{v}_1$ or $\vec{v}_2$ (such as in Fig. 1-6).

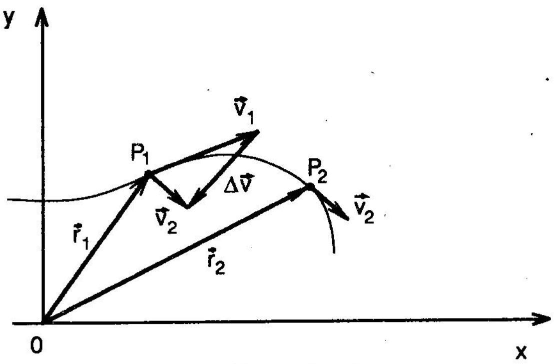

Figure 1 - 6

The vectors $\vec{v}_1$ and $\vec{v}_2$ may also have the same magnitude but different directions, and the difference of two such vectors will not be zero. Hence, acceleration can result from either a change in the magnitude of the velocity or from a change in direction of the velocity, or from a change in both.

The acceleration vector (the instantaneous acceleration vector) is defined as the limit of the average acceleration vector as the time interval Δt is allowed to approach zero:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \qquad (1\text{-}12)$$

and is thus the derivative of the velocity vector $\vec{v}$ with respect to time t or the second derivative of the position vector $\vec{r}$ with respect to time. It represents the rate of change of the velocity vector. Using components gives us:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{i} + \frac{dv_y}{dt}\vec{j} + \frac{dv_z}{dt}\vec{k} =$$

$$= \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} \qquad (1\text{-}13)$$

where

$$a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad a_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}, \quad a_z = \frac{dv_z}{dt} = \frac{d^2z}{dt^2}$$

are x , y and z components of the acceleration vector, respectively.

Note that acceleration of a motion will be nonzero not only when the magnitude of the velocity is changing but also if its direction is changing.

The magnitude of the acceleration vector can be expressed as

$$a = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_z}{dt}\right)^2} =$$

$$= \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2} . \qquad (1\text{-}14)$$

(In general, we will use the terms "velocity" and "acceleration" to mean the instantaneous values.)

1 - 5 Uniform Circular Motion

An object that moves in a circle at constant speed v is said to undergo uniform circular motion. Although the magnitude of the velocity vector remains constant in this case, its direction is continually changing. At each point the instantaneous velocity vector is in a tangent direction to the circular path (Fig. 1-7).

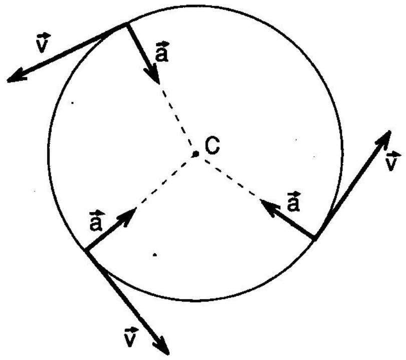

Figure 1 - 7

Since the acceleration vector is defined as the rate of change of the velocity vector,a change in direction of the velocity vector constitutes an acceleration vector. Thus an object undergoing uniform circular motion is accelerating. The acceleration of this motion is defined by

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} \qquad (1\text{-}15)$$

where $\Delta\vec{v}$ is a change in velocity vector during the short time interval Δt .

In Fig. 1-8 during time interval Δt the particle moves from the point A to the point B, covering the arc of the distance $\Delta\ell$ that subtends an angle $\Delta\varphi$. The change in the velocity vector equals $\Delta\vec{v} = \vec{v} - \vec{v}_0$. As time interval Δt approaches zero, $\Delta\ell$ and $\Delta\varphi$ approach zero, too, and $\vec{v}$ will be almost paralell to $\vec{v}_0$ and $\Delta\vec{v}$ will be essentially perpendicular to them.Thus $\Delta\vec{v}$ points toward the center of the circle. Since the acceleration $\vec{a}$, by Eq. (1-15), is in the same direction as $\Delta\vec{v}$, it must point toward the center of the circle, too (Fig. 1-7).Therefore, this acceleration is called centripetal or radial acceleration.

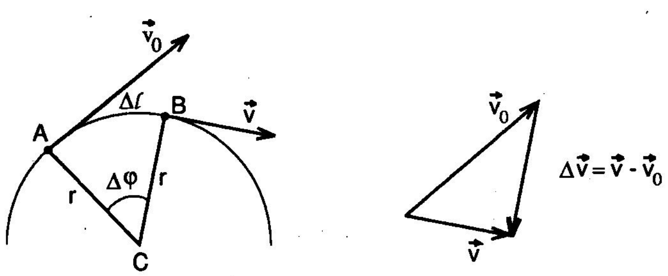

Figure 1 - 8

Now we determine the magnitude of the centripetal acceleration. The vectors $\vec{v}_o$, $\vec{v}$ and $\Delta\vec{v}$ form a triangle that is similar to the triangle ABC (Fig. 1-8). Thus we may write

$$\frac{\Delta v}{v} = \frac{\Delta \ell}{r} \quad \text{or} \quad \Delta v = \frac{v}{r}\Delta \ell .$$

To get the magnitude of the centripetal acceleration we calculate the limit

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \lim_{\Delta t \to 0} \frac{v}{r}\frac{\Delta \ell}{\Delta t} .$$

Since $\lim_{\Delta t \to 0} \frac{\Delta \ell}{\Delta t} = v$ and v and r are the constant quantities, we obtain the formula for the magnitude of the centripetal acceleration

$$a = \frac{v^2}{r} . \qquad (1\text{-}16)$$

To summarize, a particle moving in a circle of the radius r with constant speed v has an acceleration directed toward the center of the circle whose magnitude is given by Eq. (1-16). For the uniform circular motion velocity and acceleration vector are perpendicular to each other since the velocity vector $\vec{v}$ is tangential to the circle and the acceleration vector points toward the center.

1 - 6 Nonuniform Circular Motion

If the speed of a particle revolving in a circle is changing, there will be a tangential acceleration, $\vec{a}_T$, as well as the radial (centripetal) acceleration, $\vec{a}_R$. The tangential acceleration arises from the change in the magnitude of the velocity and has magnitude

$$a_T = \frac{dv}{dt} , \qquad (1\text{-}17)$$

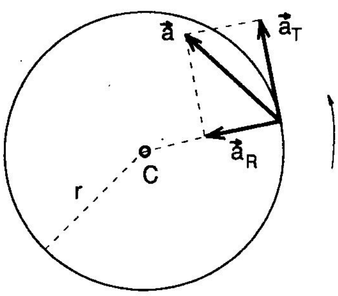

Figure 1 - 9

whereas the radial (centripetal) acceleration arises from the change in direction of the velocity and has magnitude

$$a_R = \frac{v^2}{r} . \qquad (1\text{-}18)$$

The tangential acceleration always points in a direction tangent to the circle, and is in the direction of motion (parallel to $\vec{v}$) if the speed is increasing. If the speed is decreasing, $\vec{a}_T$ points antiparallel to $\vec{v}$. In either case, $\vec{a}_T$ and $\vec{a}_R$ are always perpendicular to each other, and their directions change continually as the particle moves along its circular path. The total vector acceleration, $\vec{a}$, is the sum of these two:

$$\vec{a} = \vec{a}_T + \vec{a}_R . \qquad (1\text{-}19)$$

Since $\vec{a}_R$ and $\vec{a}_T$ are always perpendicular to each other, the magnitude of $\vec{a}$ at any moment is

$$a = \sqrt{a_T^2 + a_R^2} . \qquad (1\text{-}20)$$

1-7 Angular Variables of Circular Motion

We consider a particle rotating in a circle of radius r (see Fig. 1-10). The particle has moved along the circle a distance ℓ, and its angular position has changed by angle φ. Angle φ is given in radian. One radian (abbreviated rad) is defined as the angle subtended by arc whose length ℓ is equal to the radius r. So, in general, any angle φ is given in radians by

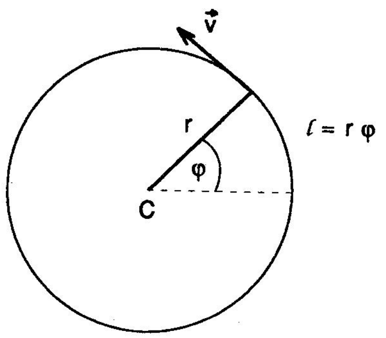

Figure 1-10

$$\varphi = \frac{\ell}{r} . \qquad (1\text{-}21)$$

We can see that the radian is dimensionless (has no units) since it is the ratio of two lengths.

Angular velocity: let φ_1 and φ_2 represent the angular positions of the particle at times t_1 and t_2, respectively. Then we define the magnitude of the average angular velocity as

$$\bar{\varphi} = \frac{\varphi_2 - \varphi_1}{t_2 - t_1} = \frac{\Delta\varphi}{\Delta t}$$

where $\Delta\varphi$ is the angular displacement during time interval Δt. The magnitude of the instantaneous angular velocity is the limit of this ratio as Δt approaches zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt} . \qquad (1\text{-}22)$$

We see that an angular velocity is defined as the rate of change of the angular displacement. The dimension of the angular velocity is s^{-1}.

Angular acceleration is defined as the change in angular velocity divided by the time required to make this change. Let ω_1 and ω_2 represent the instantaneous angular velocity at times t_1 and t_2, respectively. Then the magnitude of the average angular acceleration is defined as

$$\bar{\varepsilon} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t} .$$

So, the instantaneous angular acceleration is defined as the limit of this ratio as Δt approaches zero:

$$\varepsilon = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} . \qquad (1\text{-}23)$$

We see that an angular acceleration is defined as the rate of change of the angular velocity. The dimension of the angular acceleration is s^{-2}.

Now we can relate the angular quantities ω and ε to the linear velocity and tangential acceleration of a particle moving in a circle, respectively.

With respect to Fig. 1-10 we can write

$$v = \frac{dl}{dt} = r\frac{d\varphi}{dt} \quad \text{or} \quad v = r\omega . \qquad (1\text{-}24)$$

Thus the magnitude of the linear velocity of a particle moving in circle is equal to the radius of the circle times the magnitude of the angular velocity.

The magnitudes of the angular acceleration ε and tangential acceleration a_T are related (via Eqs. (1-17) and (1-24)):

$$a_T = \frac{dv}{dt} = r\frac{d\omega}{dt} \quad \text{or} \quad a_T = r\,\varepsilon . \qquad (1\text{-}25)$$

The radial acceleration is equal to (via Eqs. (1-18) and (1-24))

$$a_R = \frac{v^2}{r} = r\omega^2 . \qquad (1\text{-}26)$$

Sometimes it is useful to consider the frequency of rotation. By frequency f, we mean the number of revolutions per second. Since one revolution corresponds to an angle of 2π radians the frequency is given by

$$f = \frac{\omega}{2\pi} \quad \text{or} \quad \omega = 2\pi f . \qquad (1\text{-}27)$$

The dimension of the frequency is s^{-1}.

The time required for one revolution is called the period T. It is related to the frequency by

$$T = \frac{1}{f} . \qquad (1\text{-}28)$$

Example 1: The motion of a particle is described by the vector equation

$$\vec{r}(t) = (2t + 5)\,\vec{i} - t^2\,\vec{j} + \frac{1}{3}t^3\,\vec{k} \qquad [m]$$

where the parameter t is time.

Determine for any time t :

a) its coordinates and distance from the origin,
b) velocity and acceleration vectors and their magnitudes,
c) the tangential acceleration as well as the radial one.

Solution:

a) $x(t) = 2t + 5, \qquad y(t) = -t^2, \qquad z(t) = \frac{1}{3}t^3;$

these formulas represent so called parametric equations of the motion.

Distance from the origin at any time is given by the magnitude of the position vector (its absolute value), so

$$r(t) = \sqrt{x^2 + y^2 + z^2} = \sqrt{(2t+5)^2 + t^4 + \frac{1}{9}t^6} \qquad [m]$$

b)

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = 2\,\vec{i} - 2t\,\vec{j} + t^2\,\vec{k} \qquad [m.s^{-1}]$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -2\,\vec{j} + 2t\,\vec{k} \qquad [m.s^{-2}]$$

$$v(t) = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{4 + 4t^2 + t^4} = t^2 + 2 \qquad [m.s^{-1}]$$

$$a(t) = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{4 + 4t^2} = 2\sqrt{1 + t^2} \qquad [m.s^{-2}]$$

c)

$$a_T = \frac{dv}{dt} = 2t \qquad [m.s^{-2}]$$

$$a_R = \sqrt{a^2 - a_T^2} = \sqrt{4 + 4t^2 - 4t^2} = 2 \qquad [m.s^{-2}]$$

We see the radial acceleration of this motion does not depend on time. It is constant for all time of the motion.

E x a m p l e 2 : The acceleration of a motion increases at a uniform rate. The motion starts from rest and at time t_1 the magnitude of its acceleration equals a_1 .

Determine the dependence of the velocity and the trajectory of the motion on time t .

Solution: The given motion is nonuniform one, but its acceleration varies uniformly with time, thus we may write for it

$$a(t) = kt .$$

Where coefficient k can be determined from given values. Hence

$$k = \frac{a_1}{t_1} .$$

Then

$$v(t) = \int_0^t a(t)\,dt = \int_0^t \frac{a_1}{t_1} t\,dt = \frac{a_1}{2t_1} t^2 ,$$

and

$$s(t) = \int_0^t v(t)\,dt = \int_0^t \frac{a_1}{2t_1} t^2\,dt = \frac{a_1}{6t_1} t^3 .$$

E x a m p l e 3 : The angular displacement φ of a particle moving in a circle depends on time as

$$\varphi(t) = k_1 t + k_2 t^3$$

where $k_1 = \pi/10\ s^{-1}$, $k_2 = \pi/40\ s^{-3}$.

Determine the dependence of the angular velocity ω as well as the angular acceleration ε of the motion on time t .

Solution: By using the definitions for ω and ε we can write:

$$\omega(t) = \frac{d\varphi}{dt} = k_1 + 3k_2 t^2$$

$$\varepsilon(t) = \frac{d\omega}{dt} = 6k_2 t$$

E x a m p l e 4 : A particle moves uniformly along the circle of the radius r with an angular velocity ω.

Give an answer on the following questions: what are its position vector, the velocity vector, the acceleration vector, the magnitudes of the radial and tangential acceleration.

Solution: The coordinates of a moving particle are given by

$$x = r\cos\omega t , \qquad y = r\sin\omega t .$$

So, the position vector of a particle equals

$$\vec{r}(t) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = r(\vec{i}\cos\omega t + \vec{j}\sin\omega t) .$$

Then, the velocity vector equals

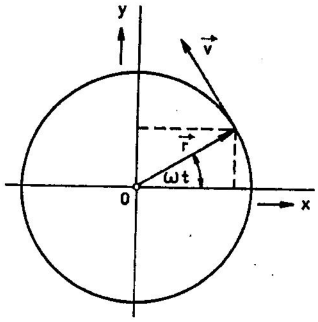

Figure 1-11

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -\vec{i}\, r\,\omega \sin\omega t + \vec{j}\, r\,\omega \cos\omega t \;.$$

And the acceleration vector equals

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = -\vec{i}\, r\,\omega^2 \cos\omega t - \vec{j}\, r\,\omega^2 \sin\omega t =$$

$$= -r\,\omega^2(\vec{i}\cos\omega t + \vec{j}\sin\omega t) = -\omega^2\vec{r}\;.$$

It is clear that the acceleration vector has an opposite direction than the position vector, that is, it points toward the center and thus it is identical with the vector of the centripetal acceleration. So, we have $\vec{a} \equiv \vec{a}_R$. Because of $\vec{a} = \vec{a}_T + \vec{a}_R$, then the vector of the tangential acceleration must be equal to zero, $\vec{a}_T = 0$.
So, the magnitude of the total acceleration equals that of the centripetal acceleration.
Hence

$$a = a_R = |\vec{a}| = \omega^2 r\;.$$

E x a m p l e 5 : A flywheel rotates with frequency $n = 1500$ revolutions per minute. Due to braking its motion becomes an uniformly retarded one and it finishes during time $t_o = 30$ seconds after the braking started.

Determine the angular acceleration and the number of revolutions performed from the beginning of braking till stop of the motion.

Solution: An instantaneous value of an angular velocity of the uniformly accelerated circular motion is given by

$$\omega(t) = \omega_o + \varepsilon t$$

where ω_o is an initial angular velocity of this motion. For our case it equals

$$\omega_o = 2\pi f = 2\pi\,\frac{1500}{60} = 50\,\pi\, s^{-1}\;.$$

For time t_o $\omega(t)$ must be zero, so

$$\omega(t_o) = \omega_o + \varepsilon t_o = 0\;.$$

Thus an angular acceleration equals

$$\varepsilon = -\frac{\omega_o}{t_o} = -\frac{50\,\pi}{30} = -\frac{5}{3}\,\pi\, s^{-2}\;.$$

The angle subtended during time t_o equals

$$\varphi_o = \int_0^{t_o} \omega(t)\, dt = \int_0^{t_o} (\omega_o + \varepsilon t)\, dt = \omega_o t_o + \frac{1}{2}\varepsilon t_o^2 =$$

$$= 50.\pi.30 - \frac{5}{6}\pi.900 = 1500\,\pi - 750\,\pi = 750\,\pi\;.$$

The number of revolutions performed during time $t_o = 30$ s equals

$$N = \frac{\varphi_o}{2\pi} = \frac{750\,\pi}{2\,\pi} = 375.$$

Example 6: A disc wheel starts to rotate from rest with constant angular acceleration. At time $t = 20$ s its frequency equals 200 revolutions per minute.

Determine the angular acceleration of the motion. How many revolutions are performed during this time?

Solution: The angular velocity of the wheel equals $\omega = 2\pi n$, where $n = \frac{200}{60}$ represents the number of revolutions per second.

So, the angular velocity at time $t = 20$ s equals

$$\omega = 2\pi n = \frac{2\pi.200}{60} \doteq 21\ s^{-1}.$$

Since the motion starts from rest the angular acceleration will be equal to

$$\varepsilon = \frac{\omega}{t} \doteq \frac{21}{20} = 1{,}05\ s^{-2}.$$

The angle subtended during time $t = 20$ s equals

$$\varphi(t) = \int_0^t \omega(t)\, dt = \int_0^t \varepsilon t\, dt = \frac{1}{2}\varepsilon t^2 = \frac{1}{2}\, 1{,}05.20^2 = 210.$$

So, the number of revolutions during time $t = 20$ s equals

$$N = \frac{210}{2\pi} \doteq 33{,}4.$$

2. DYNAMICS

2 - 1 The First Law of Motion

In Chapter 1 we have discussed how motion is described in terms of velocity and acceleration. In this Chapter we want to deal with the question of why objects move as they do, what causes a body to accelerate or decelerate. We can answer that a force is required and therefore we investigate the connection between force and motion. A force has direction as well as magnitude and is a vector that follows the rules for vectors. We represent any force on a diagram by an arrow and its length is drawn proportional to magnitude of the force.

Newton's analysis of motion is summarized into three laws of motion. The first law of motion states that

> every body continues in its state of rest or of uniform speed in a straight line unless it is compelled to change that state by forces acting on it.

The tendency of a body to maintain its state of rest or of uniform motion in a straight line is called inertia. As a result, the first law of motion is often called the law of inertia.